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REFLECTION AND TRANSMISSION OF ELECTROMAGNETIC WAVES BY THIN CURVED SHELLS

by

JOSEPH BISHOP KELLER

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ABSTRACT

In this paper the general problem of scattering of an arbitrary incident electromagnetic field by a conducting or non-conducting obstacle is investigated. The partial differential equations and boundary conditions satisfied by the field are transformed into a pair of inhomogeneous linear integro-differential equations for two vector functions. For an obstacle which is a thin shell of constant thickness h, a formal procedure for obtaining a solution of these equations as power series in h is given. The lowest order term in this solution is the incident field. An explicit expression for the next term is found in the form of a surface integral. This integral is evaluated approximately by an extension to surface integrals of the method of stationary phase. The scattered field, given by the terms so obtained, is that which is reflected, diffracted or transmitted by a thin shell. The physical properties of this solution are examined in detail, and satisfactory agreement is found with many results previously obtained by other methods.

CONTENTS

Article	Page
1. Introduction	1
2. Derivation of the Integral Equation	6
3. Reduction of the Equations for an Obstacle which is a Thin Shell	15
4. Evaluation of $E_{(1)}(r)$ and $H_{(1)}(r)$	21
5. Properties of the Solution	26
Geometrical Factor	26
Phase Factor	30
Reflectivity Factor	34
6. Conclusion	35
Appendix I	38
Appendix II	44
Bibliography	48
List of Symbols	50

1. Introduction

The problem of reflection and transmission of electromagnetic waves by an obstacle is that of determining the electromagnetic field in a region containing a source of radiation and an obstacle with electrical properties different from those of the surrounding medium. The solution of this problem can be expressed as the sum of two parts, the incident and scattered fields. The incident field is that field which would be present in the absence of the obstacle (i.e. obstacle replaced by the surrounding medium); the scattered field is the difference between the total field and the incident field, and is due to the presence of the obstacle. The scattered field is also called the reflected, transmitted, or diffracted field, depending upon the region of space in which it is being discussed.

Often the properties of the source are altered by the reflected field. This effect (change in radiation impedance of the source) is of importance when the reflected field is comparable in magnitude to the incident field at the source. In the present investigation it will not be considered, since the reflected fields will be much weaker at the source than the incident field. Furthermore, it will be assumed that the region under consideration is unbounded and that the medium is homogeneous and isotropic, although the methods are applicable to more general problems. In addition it will be assumed that the incident field is known.

In order to solve the problem it is necessary to obtain, in the obstacle as well as in the surrounding medium, a solution of Maxwell's equations which satisfies the boundary conditions on the obstacle boundary. This solution must also contain the incident field. For incident plane or spherical waves this last condition can be fulfilled by requiring the solution to have an appropriate singularity at the source; in other cases this condition may be more troublesome.

Because of the mathematical complexity of the problem, various methods of approximations have been devised. Among these the method of geometrical optics and the Kirchhoff method in the theory of diffraction are probably most widely used. Recently R. K. Luneberg (10)* has shown (by the method of characteristics) that the geometrical optics result is approached by the solution of Maxwell's equations as the wave length tends to zero, for arbitrary surfaces of discontinuity in the medium.

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* Numbers in parentheses refer to the bibliography at the end of the paper.

But long before this, geometrical optics was used for tracing reflected and refracted rays and deriving the lens and mirror laws. More recently the relative intensity at any point in the field reflected from an arbitrary curved surface has been obtained exactly under the assumptions used in geometrical optics (for incident plane or spherical waves) (12)(2) and the method can be extended to the transmitted field and other problems.

The geometrical solution has several limitations. In the first place, it takes no account of the physical properties of the reflecting surface and therefore can yield only relative intensities in the reflected field, i.e., the ratio of the reflected intensity at one point to that at another. Secondly, it takes no account of the incident wave length and therefore does not yield diffraction effects. Thirdly, it leads to infinite intensities at foci. In spite of these objections, the geometrical solution yields useful information about the reflected field, especially for short wave lengths.

The second and third objections mentioned above may be overcome by using the Kirchhoff method, which has been widely used in optical diffraction problems but has been little used in reflection problems until lately. It consists in expressing the field as an integral involving the field and its derivatives on the obstacle surface, and then assuming values for these quantities on this surface. [(1) p 70]. Until now the only justification for this procedure has been that its results agree with experiment. It is believed that the results of the present paper provide some theoretical justification for it in some problems (see below).

Rayleigh [(13) pp 118-126] proposed the use of the Kirchhoff method in problems of acoustic reflection, and several such problems have been solved in this way by Willis (16). Willis evaluated the integrals by making use of the Fresnel zone concept. In the electromagnetic case the Kirchhoff method has been applied to reflection from several objects of special shape at the University of Pennsylvania (17)*. Finally the reflection of acoustic or electromagnetic waves from an arbitrary curved surface has been treated approximately by the Kirchhoff method (12)(7). The solutions in the two cases are identical, and yield for the reflected amplitude,

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* In the evaluation of the Kirchhoff integral in ref. (17) only linear terms are retained in the expansion of the phase. This is essentially the Frauenhofer approximation of diffraction theory. While this facilitates the evaluation of the integral some of the results are physically unreasonable. The retention of the terms of second degree, in the present paper and in ref. (7), which is the Fresnel approximation, eliminates these discrepancies.

the incident amplitude multiplied by two factors. One factor is the same as the reflection coefficient obtained by geometrical optics (in directions of specular reflection) and is called the geometrical factor; the other is called the phase factor. The phase factor approaches unity in directions of specular reflection and zero otherwise, as the ratio of the wave length to the obstacle dimensions approaches zero, and generally has a maximum in the specular directions for any wave length. Thus an explicit demonstration of the law of reflection for curved surfaces is obtained. The phase factor gives the result a dependence on the obstacle size as well as the wave length. At foci the amplitude remains finite, overcoming another defect of the geometrical result.

The Kirchhoff method also has defects. First, it has no theoretical basis. Secondly, it takes practically no account of the physical properties of the obstacle and therefore yields only relative amplitudes in the reflected field and very little about its polarization. It yields more information about the reflected and diffracted fields however, than does the geometrical solution.

In order to overcome the defects of the Kirchhoff solution it is necessary to solve Maxwell's equations inside as well as outside the obstacle. Several special problems have been solved in which this has been done. Probably the first was that of reflection of normally incident plane waves from the plane interface between two semi-infinite media or from an infinite plate of finite thickness (15). The corresponding problem of acoustics was solved long ago for any angle of incidence [ref. (13) pp 86,87], but apparently this had not been done for the plate of finite thickness in the electromagnetic case until recently. The solution of this problem has been given by R. K. Luneberg, who has also treated the case of several parallel plates and of plates with variable properties (11). An approximate solution of the acoustical problem for many parallel plates or for plates of variable properties has been given by M. E. Rose (14), and an exact solution for a special case of the plate with variable acoustical properties by Courant and Friedrichs (5).

Exact solutions have also been obtained in the acoustic and electromagnetic cases for a point source or point dipole at the center of a spherical shell (8) (9). Similar results can be obtained for a line source on the axis of a cylindrical shell.

The problem of scattering or diffraction of plane or spherical waves incident on a sphere or circular cylinder has also been discussed by many authors in both the acoustic [(13) pp 236-285] and electromagnetic cases [(15) p 563]. The

solutions are usually given in series form, and these are very slowly convergent for wave lengths small compared to the radius of the sphere or cylinder. If the wave length is large compared to the radius, in the case of a sphere, agreement with the Rayleigh scattering law is obtained. The Rayleigh scattering law gives the field at large distances scattered from an object small compared to the wave length obtained by Rayleigh for sound. The corresponding electromagnetic problem has also been treated [(15) p. 572].

In the acoustic case the reflection of plane or spherical waves from an arbitrarily curved thin shell has been solved approximately (12), taking account of propagation inside the shell, without making assumptions about boundary values, but making use of the usual continuity conditions at the boundary. This was done by transforming the wave equation to an integral equation which was solved approximately. The solution for the reflected sound field consists of the incident pressure multiplied by three factors - a geometrical factor, a phase factor, and a reflectivity factor. The first factor coincides with the result of geometrical acoustics or optics in direction of specular reflection, the first two factors coincide with the result of the Kirchhoff theory, and the third (reflectivity) factor coincides with the reflection coefficient of a thin infinite plate except for a term involving the curvature of the surface.

In this paper, the same method is applied to the electromagnetic problem, and similar results are obtained. The geometrical and phase factors, which are the same for all components, agree exactly with the corresponding factors in the acoustic solution. Since the geometrical and Kirchhoff solutions are identical in the acoustic and electromagnetic cases (see page30) this agreement might have been expected. Thus all properties of the solutions which depend only upon the first two factors will be the same in the acoustic and electromagnetic cases. It should first be mentioned that the solution provides a partial justification for the Kirchhoff theory as well as for the use of geometrical optics or acoustics. Furthermore, from the geometrical factor the mirror law and conditions for the formation of point images or foci can be obtained. From the phase factor the law of reflection for curved surfaces, the approach to geometrical optics in the limit as the wave length tends to zero and the change of phase at a focus or point image can be derived. In addition, in ref. (12) the Kirchhoff theory was applied to refraction at the curved interface between two media and the results apply to the electromagnetic as well as the acoustic case. Snell's law for curved surfaces, geometrical optics in the limit of zero wave length, the lens law, and conditions for the formation

of point images or foci, as well as the change of phase at a focus, are some of the results obtainable from the solution.

The reflectivity factors, on the other hand, are not the same for all components of the electromagnetic field. They are equal to the reflection coefficients obtained for plane waves incident obliquely on a thin infinite plate except for a term involving the curvature of the surface. Thus, except for this additional term considerations of the polarization of the reflected wave would be the same as for the case of plane waves incident on a thin infinite plate.

The reflection of electromagnetic waves from thick shells or obstacles can be treated by the successive approximation method formulated in this paper. Since the calculations required are quite elaborate, however, an alternative method of treating such problems, which has been successfully employed in acoustics, will be described (see page 36).

The method of attack in this paper is based upon a transformation of the partial differential equations satisfied by the electromagnetic field into an integro-differential equation. In section 2 this equation will be derived for an arbitrary obstacle in the field. We shall then simplify the equation in section 3 by restricting our considerations to an obstacle in the form of a thin shell with constant thickness. The remainder of the paper will be devoted to the approximate solution of the problem, in the case of the thin shell, and the discussion of this solution.

2. Derivation of the Integral Equation

We shall suppose that an obstacle of one material is immersed in an infinite medium of another material. We shall designate the infinite medium as well as its material by (a), and likewise, the obstacle and its material will both be called (b). Instead of an obstacle in a medium it will be convenient to think of this situation as one in which one medium is embedded in another. Each medium is assumed to be homogeneous and isotropic and may be either a dielectric or a conductor, i.e., the conductivity can be zero or have any finite value.

A periodic source of electromagnetic waves, not necessarily a point source, is located in medium (a), and the fields arising from this source are assumed to be proportional to $e^{-i\omega t}$. (For the treatment of other types of time dependence the method of harmonic analysis may be employed since the problem is linear.) Under these conditions it can be shown [(15) p 392] that the total electric field E satisfies the homogeneous wave equation

$$(\nabla^2 + k_a^2)\ E(r) = 0 \text{ outside the source and the obstacle,} \qquad (1)$$

$$(\nabla^2 + k_b^2)\ E(r) = 0 \text{ inside the obstacle.} \qquad (2)$$

In these equations k_a and k_b denote the values of k in medium (a) and (b), respectively; $k = \frac{\omega^2}{c^2}\ \varepsilon'\ \mu$, where $\varepsilon' = \varepsilon + \frac{4\pi i \sigma}{\omega}$. Similar equations hold for H(r), i.e., equations (1) and (2) apply if E(r) is replaced by H(r). At the boundary between (a) and (b) appropriate continuity conditions must be satisfied; these conditions will be discussed later.

Equations (1) and (2) can be transformed into a single integro-differential equation in the following way. Let $G(r,r')$ be a function of r and r' satisfying the equation $(\nabla^2 + k_a^2)\ G(r,r') = 0$, with respect to the second variable r', throughout space except at r' = r where it has a singularity of the form $\frac{1}{|r - r'|}$. Later we shall take $G(r,r') = \frac{e^{ik_a|r - r'|}}{|r - r'|}$, the Green's function for infinite space. Now, by Green's theorem, if Q and G are finite, single-valued, differentiable functions in a three-dimensional region V bounded by a sectionally smooth surface S (i.e., consisting of a finite number of parts, each with continuously turning tangent planes), and n represents the normal to S into V, then [(4) p 390]

$$\int_S \left(Q \frac{\partial G}{\partial n} - G \frac{\partial Q}{\partial n}\right) dS = \int_V (Q \nabla^2 G - G \nabla^2 Q)\, dV. \qquad (3)$$

This equation is still valid if Q is a vector function, since it holds for each component separately. It is understood that r' is the variable of integration.

We now set $Q = E - E_o$, where E_o represents the electric field in the absence of the obstacle, i.e., the incident field, and observe that E_o satisfies the equation $(\nabla^2 + k_a^2) E_o = 0$ throughout space except at the source. We assume that $E - E_o$, which is the scattered field, has no singularities. Since E satisfies equations (1) and (2),

$$Q \nabla^2 G - G \nabla^2 Q = - k_a^2 (E - E_o) G + k_a^2 (E - E_o) G = 0,$$

if r' is in (a),

$$= - k_a^2 (E - E_o) G + k_b^2 EG - k_a^2 E_o G$$

$$= (k_b^2 - k_a^2) EG,$$

if r' is in (b).

From equation (3) and the above results it follows that, if the region $\bar{V}$ bounded by $\bar{S}$ is in medium (a) and the normal is directed into $\bar{V}$,

$$\int_{\bar{S}} \left((E - E_o) \frac{\partial G}{\partial n} - G \frac{\partial (E - E_o)}{\partial n}\right) d\bar{S} = 0. \qquad (4)$$

Similarly, if region V bounded by S is in medium (b) and the normal is directed into V,

$$\int_S \left((E - E_o) \frac{\partial G}{\partial n} - G \frac{\partial (E - E_o)}{\partial n}\right) dS = (k_b^2 - k_a^2) \int_V EG\, dV. \qquad (5)$$

We now apply equation (4) to the region $\bar{V}$ bounded by the obstacle surface S', a sphere at infinity σ_∞, and a sphere σ_r around the singularity of G at $r' = r$ if r lies in medium (a). This choice of $\bar{V}$ insures that neither $E - E_o$ nor G has a singularity in $\bar{V}$. We assume, furthermore, that both $E - E_o$ and G behave like outgoing waves at infinity, and this insures the vanishing of

the integral over σ_∞ [(1) p 25]. The integral $\int_{\bar{S}} \left((E-E_o) \frac{\partial G}{\partial n} - G \frac{\partial (E-E_o)}{\partial n} \right) d\bar{S}$ thus consists of two parts corresponding to σ_r and S'. The integral over σ_r is equal to $-4\pi \left(E(r) - E_o(r) \right)$ [(1) p 24], and hence we have from (4),

$$E(r) - E_o(r) = \frac{1}{4\pi} \int_{S'} \left\{ \left(E(r') - E_o(r') \right) \frac{\partial G(r,r')}{\partial n} - G \frac{\partial \left(E(r') - E_o(r') \right)}{\partial n} \right\} dS, \qquad (6a)$$

if r is in (a),

$$0 = \frac{1}{4\pi} \int_{S'} \left\{ \left(E(r') - E_o(r') \right) \frac{\partial G(r,r')}{\partial n} - G \frac{\partial \left(E(r') - E_o(r') \right)}{\partial n} \right\} dS, \qquad (6b)$$

if r is in (b).

Similarly, applying equation (5) to the obstacle volume from which the point r is excluded by a sphere σ_r if r is in (b), we obtain

$$E(r) - E_o(r) = \frac{1}{4\pi} \int_{S'} \left\{ \left(E(r') - E_o(r') \right) \frac{\partial G(r,r')}{\partial n} - G \frac{\partial \left(E(r') - E_o(r') \right)}{\partial n} \right\} dS + \frac{k_b^2 - k_a^2}{4\pi} \int_V E\, G\, dV, \qquad (7a)$$

if r is in (b),

$$0 = \frac{1}{4\pi} \int_{S'} \left\{ \left(E(r') - E_o(r') \right) \frac{\partial G(r,r')}{\partial n} - G \frac{\partial \left(E(r') - E_o(r') \right)}{\partial n} \right\} dS + \frac{k_b^2 - k_a^2}{4\pi} \int_V E\, G\, dV, \qquad (7b)$$

if r is in (a)

The variable of integration in each of the integrals in equations (6a)-(7b) is r' .

In equations (7a) and (7b) the normal direction is into (b) and hence is opposite to that used in (6a) and (6b) which is into (a). Changing the normal direction in (7a) and (7b), and then adding, (6a) and (7b), and (6b) and (7a), we have for r in either (a) or (b),*

$$E(r) - E_o(r) = \frac{1}{4\pi} \int_{S'} \left\{ (E^a - E^b) \frac{\partial G}{\partial n} - G \left(\frac{\partial E^a}{\partial n} - \frac{\partial E^b}{\partial n} \right) \right\} dS + \frac{k_b^2 - k_a^2}{4\pi} \int_V E\, G\, dV, \qquad (8)$$

in which E^a and E^b represent the values of E on the (a) and (b) sides of S', respectively.

In precisely the same way we obtain for the magnetic field

$$H(r) - H_o(r) = \frac{1}{4\pi} \int_{S'} \left\{ (H^a - H^b) \frac{\partial G}{\partial n} - G \left(\frac{\partial H^a}{\partial n} - \frac{\partial H^b}{\partial n} \right) \right\} dS + \frac{k_b^2 - k_a^2}{4\pi} \int_V H\, G\, dV. \qquad (8')$$

If the point r is on the surface S' it is excluded by a sphere half of which lies in (a) and half in (b). In this case each of equations (6a)-(7b) has the left-hand side equal to $\frac{1}{2} \left\{ E(r) - E_o(r) \right\}$. When the additions are performed, as described above, we still obtain equation (8) (or (8')). Thus equations (8) and (8') hold for r anywhere.

If the surface S' and the volume V vanish, i.e., no obstacle is present, $E(r) = E_o(r)$ (and $H(r) = H_o(r)$). Thus the solution in the absence of the obstacle reduces to the incident field, as expected. The two integrals in (8) (and (8')) therefore give us the reflected, scattered, or disturbed field.

In order to evaluate the differences occurring in the integrand of the surface integral in (8) (and (8')) we may use Maxwell's equations and the continuity conditions that must apply to the fields at the interface between the two media. If n and t indicate normal and tangential components, respectively, the continuity conditions are the following [(15) pp 34-38, 483]:

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* See ref. [12] equation (7). This equation is the analogue of equation (8) in the present paper, except that (7) already includes boundary conditions.

$$E_t^a = E_t^b \,, \quad \varepsilon_a' E_n^a = \varepsilon_b' E_n^b \,, \quad \left(\varepsilon' = \varepsilon + \frac{4\pi i\sigma}{\omega}\right) \tag{9}$$

$$H_t^a = H_t^b \,, \quad \mu_a H_n^a = \mu_b H_n^b \,. \tag{10}$$

Since $\vec{E}^a = \vec{E}_t^a + \vec{E}_n^a$ and $\vec{E}^b = \vec{E}_t^b + \vec{E}_n^b$, we obtain from (9),

$$\vec{E}^a - \vec{E}^b = \left(\frac{\varepsilon_b'}{\varepsilon_a'} - 1\right) E_n^b \vec{n}, \tag{11}$$

where $\vec{n}$ is the unit normal.

Similarly, from (10),

$$\vec{H}^a - \vec{H}^b = \left(\frac{\mu_b}{\mu_a} - 1\right) H_n^b \vec{n} \,. \tag{12}$$

The value of $E^a - E^b$ from (11) (and $H^a - H^b$ from (12)) can be substituted into (8) and (8').

The evaluation of the difference $\frac{\partial E^a}{\partial n} - \frac{\partial E^b}{\partial n}$ is more difficult and requires the use of Maxwell's equations. For this purpose we set up an orthogonal coordinate system in a neighborhood of the surface S'. In this system the coordinate surfaces are the surface S' and its parallel surfaces, and the surfaces generated by the normals to the lines of curvature, u_1 = const. and u_2 = const. on S'. The three coordinates of a point are then the distance n from S' along a normal and the values of u_1 and u_2 corresponding to the intersection of the normal and S'. The positive direction on n is toward the medium (a). In the neighborhood of the surface S' this coordinate system is orthogonal, and if S' is given by the vector $x(u_1, u_2)$ having continuous partial derivatives, it is single-valued [(6) p 272]. The square of the differential of arc-length may be written

$$ds^2 = dn^2 + g_1 du_1^2 + g_2 d u_2^2 \,, \tag{13}$$

where g_1 and g_2 are functions of n, u_1, and u_2 [(15) p 38]. If n, i_1 and i_2 represent unit vectors along n, u_1, and u_2, respectively, the field vectors may be written

$$E = E_n n + E_1 i_1 + E_2 i_2 \,, \qquad H = H_n n + H_1 i_1 + H_2 i_2 \,. \tag{14}$$

Since it is assumed that there is no free charge present, and the fields are proportional to $e^{-i\omega t}$, Maxwell's equations are

$$\nabla \cdot E = 0, \qquad \nabla \cdot H = 0, \tag{15}$$

$$\nabla \times E = \frac{i\omega\mu}{c} H, \qquad \nabla \times H = \left(\frac{4\pi\sigma}{c} - \frac{i\omega\varepsilon}{c}\right) E = -\frac{i\omega\varepsilon'}{c} E. \tag{16}$$

It is understood that the differential operators in these equations are expressed in terms of the coordinates n, u_1, and u_2 [(15) p 49].

We are now in a position to derive expressions for the differences in the normal derivatives of the components of E and H as we cross the surface S'. Using the first of equations (14) with the first of equations (15) we have for the electric vector,

$$\frac{1}{\sqrt{g_1 g_2}} \left[\frac{\partial}{\partial n} (E_n \sqrt{g_1 g_2}) + \frac{\partial}{\partial u_1} (E_1 \sqrt{g_2}) + \frac{\partial}{\partial u_2} (E_2 \sqrt{g_1}) \right] = 0. \tag{17}$$

The left-hand side of equation (17) is the expression for the divergence of E in general orthogonal coordinates. In order to find the value of $\frac{\partial E^b}{\partial n} - \frac{\partial E^a}{\partial n}$ on the surface we employ equation (17) first on the (b) side of S' and then on the (a) side and subtract. Applying this step, and carrying out the differentiation $\frac{\partial}{\partial n}(E_n \sqrt{g_1 g_2})$ only, we obtain

$$\frac{\partial E_n^b}{\partial n} - \frac{\partial E_n^a}{\partial n} = \frac{1}{\sqrt{g_1 g_2}} \left[E_n^a \frac{\partial}{\partial n} \sqrt{g_1 g_2} + \frac{\partial}{\partial u_1} (E_1^a \sqrt{g_2}) + \frac{\partial}{\partial u_2} (E_2^a \sqrt{g_1}) \right.$$

$$\left. - E_n^b \frac{\partial}{\partial n} \sqrt{g_1 g_2} - \frac{\partial}{\partial u_1} (E_1^b \sqrt{g_2}) - \frac{\partial}{\partial u_2} (E_2^b \sqrt{g_1}) \right]. \tag{18}$$

It follows from the first of equations (9) that $E_1^a = E_1^b$ and

$E_2^a = E_2^b$ on S'. Hence equation (18) becomes, at any point $(0, u_1, u_2)$ on S',

$$\frac{\partial E_n^b}{\partial n} - \frac{\partial E_n^a}{\partial n} = (E_n^a - E_n^b) \frac{\frac{\partial}{\partial n} \sqrt{g_1 g_2}}{\sqrt{g_1 g_2}}.$$

Now $\frac{\partial}{\partial n}\sqrt{g_1 g_2} = 2\, G_m \sqrt{g_1 g_2}$ where G_m denotes the mean curvature of the surface at the point $(0, u_1, u_2)$ [ref.(3) § 105]. Using equation (11) and the fact that $E_t^a = E_t^b$, it follows that

$$\frac{\partial E_n^b}{\partial n} - \frac{\partial E_n^a}{\partial n} = \left(\frac{\varepsilon'_b}{\varepsilon'_a} - 1\right) E_n^b \cdot 2G_m . \qquad (19)$$

(The mean curvature G_m is positive if the center of curvature is opposite that side of the surface which agrees with the positive direction of the normal.)

In exactly the same way we can obtain for the normal component of the magnetic vector, using the second of equations (15) and the second of equations (14) together with equation (12),

$$\frac{\partial H_n^b}{\partial n} - \frac{\partial H_n^a}{\partial n} = \left(\frac{\mu_b}{\mu_a} - 1\right) H_n^b \cdot 2G_m . \qquad (20)$$

In order to obtain an expression for the difference of the normal derivatives of the tangential component E_1 on the two sides of the surface S', we use the first of equations (16) with the first of equations (14) and equate the u_2-components. For $\nabla \times E$ we have, [(15) p 49],

$$\nabla \times E = \frac{1}{\sqrt{g_1 g_2}}\left[\frac{\partial}{\partial u_1} E_2 \sqrt{g_2} - \frac{\partial}{\partial u_2} E_1 \sqrt{g_1}\right] n$$

$$+ \frac{1}{\sqrt{g_2}}\left[\frac{\partial}{\partial u_2} E_n - \frac{\partial}{\partial n} E_2 \sqrt{g_2}\right] i_1 + \frac{1}{\sqrt{g_1}}\left[\frac{\partial}{\partial n} E_1 \sqrt{g_1} - \frac{\partial}{\partial u_1} E_n\right] i_2$$

Since $\nabla \times E - \frac{i\omega\mu}{c} H = 0$ on both the (a) and (b) sides of the surface S', we have on equating u_2-components,

$$\frac{1}{\sqrt{g_1}}\left[\frac{\partial}{\partial n}(E_1^b \sqrt{g_1}) - \frac{\partial}{\partial u_1}(E_n^b)\right] - \frac{i\omega\mu_b}{c} H_2^b$$

$$= \frac{1}{\sqrt{g_1}}\left[\frac{\partial}{\partial n}(E_1^a \sqrt{g_1}) - \frac{\partial}{\partial u_1}(E_n^a)\right] - \frac{i\omega\mu_a}{c} H_2^a = 0. \qquad (21)$$

Since, from (9) and (10), $E_1^b = E_1^a$ and $H_2^b = H_2^a$, equation (21) yields

$$\frac{\partial E_1^b}{\partial n} - \frac{\partial E_1^a}{\partial n} = \frac{i\omega}{c}(\mu_b - \mu_a)\, H_2^b + \frac{1}{\sqrt{\varepsilon_1}}\left(\frac{\partial E_n^b}{\partial u_1} - \frac{\partial E_n^a}{\partial u_1}\right) . \qquad (22)$$

Thus, while the component E_1 is continuous across the surface the normal derivative $\frac{\partial E_1}{\partial n}$ is not. From equation (11) and the fact that $E_t^a = E_t^b$ the last term of (22) becomes $\frac{1}{\sqrt{\varepsilon_1}}\left(1 - \frac{\varepsilon'_b}{\varepsilon'_a}\right) \frac{\partial E_n^b}{\partial u_1}$. Hence

$$\frac{\partial E_1^b}{\partial n} - \frac{\partial E_1^a}{\partial n} = \frac{i\omega}{c}(\mu_b - \mu_a)\, H_2^b + \frac{1}{\sqrt{\varepsilon_1}}\left(1 - \frac{\varepsilon'_b}{\varepsilon'_a}\right) \frac{\partial E_n^b}{\partial u_1} . \qquad (23)$$

In exactly the same way, equating u_1-components, we obtain

$$\frac{\partial E_2^b}{\partial n} - \frac{\partial E_2^a}{\partial n} = -\frac{i\omega}{c}(\mu_b - \mu_a)\, H_1^b + \frac{1}{\sqrt{\varepsilon_2}}\left(1 - \frac{\varepsilon'_b}{\varepsilon'_a}\right) \frac{\partial E_n^b}{\partial u_2} . \qquad (24)$$

Similarly, using the second of equations (9), (10), (11), and (16), respectively, we obtain for the normal derivatives of the tangential components of H,

$$\frac{\partial H_1^b}{\partial n} - \frac{\partial H_1^a}{\partial n} = -\frac{i\omega}{c}(\varepsilon'_b - \varepsilon'_a)\, E_2^b + \frac{1}{\sqrt{\varepsilon_1}}\left(1 - \frac{\mu_b}{\mu_a}\right) \frac{\partial H_n^b}{\partial u_1} , \qquad (25)$$

$$\frac{\partial H_2^b}{\partial n} - \frac{\partial H_2^a}{\partial n} = \frac{i\omega}{c}(\varepsilon'_b - \varepsilon'_a)\, E_1^b + \frac{1}{\sqrt{\varepsilon_2}}\left(1 - \frac{\mu_b}{\mu_a}\right) \frac{\partial H_n^b}{\partial u_2} . \qquad (26)$$

Inserting into equation (8) the expressions for the differences that appear in the surface integral, given by equations (11),(19),(23), and (24), with the further assumption that the scales on u_1 and u_2 are so chosen that $g_1 = g_2 = 1$

on S', we obtain for the electric vector

$$E(r) = E_o(r) + \frac{1}{4\pi}\int_{S'} \left\{ \left(\frac{\varepsilon'_b}{\varepsilon'_a} - 1\right) E_n^b \left(\frac{\partial G}{\partial n} + 2GG_m\right)m \right.$$

$$\left. + G\left[\frac{i\omega}{c}(\mu_b - \mu_a)(i_1 H_2^b - i_2 H_1^b) + \left(1 - \frac{\varepsilon'_b}{\varepsilon'_a}\right)\left(i_1 \frac{\partial E_n^b}{\partial u_1} + i_2 \frac{\partial E_n^b}{\partial u_2}\right)\right]\right\} dS$$

$$+ \frac{k_b^2 - k_a^2}{4\pi}\int_V EG \; dV. \qquad (27)$$

Similarly, substituting from equations (12), (20), (25), and (26), we obtain

$$H(r) = H_o(r) + \frac{1}{4\pi}\int_{S'} \left\{ \left(\frac{\mu_b}{\mu_a} - 1\right) H_n^b \left(\frac{\partial G}{\partial n} + 2GG_m\right)n \right.$$

$$\left. + G\left[\frac{-i\omega}{c}(\varepsilon'_b - \varepsilon'_a)(i_1 E_2^b - i_2 E_1^b) + \left(1 - \frac{\mu_b}{\mu_a}\right)\left(i_1 \frac{\partial H_n^b}{\partial u_1} + i_2 \frac{\partial H_n^b}{\partial u_2}\right)\right]\right\} dS$$

$$+ \frac{k_b^2 - k_a^2}{4\pi}\int_V HG \; dV \qquad (28)$$

Equations (27) and (28) are the fundamental integro-differential equations for the fields in the presence of an obstacle, and are valid for all values of r. From these equations it can be seen that if $\varepsilon'_b = \varepsilon'_a$ and $\mu_b = \mu_a$ (in which case $k_b = k_a$), the integrals vanish and the fields reduce to the incident fields, as expected. A formal procedure, namely a method of iteration, can be used for obtaining solutions of these equations. One may use the incident field as a first approximation. Using this first approximation in the integrals, a second approximation can be obtained; etc. A modification of this method, will be used in the following section to obtain an approximate solution of the problem when the obstacle is a thin shell.

3. Reduction of the Equations for an Obstacle which is a Thin Shell.

The integral equations (27) and (28) may be simplified, if the obstacle is a thin shell of constant thickness, by transforming the volume integrals into surface integrals. In this case we may think of the surface S' as consisting of the two parallel faces S_I and S_{II}, neglecting the thin edges of the obstacle which contribute little to the total. The surface integrals in equations (27) and (28) then extend over S_I and S_{II}.

The integrand at a point r'_{II} on S_{II} can be expressed by a Taylor's series in terms of its normal derivatives at the point r'_I on S_I which is on the normal passing through r'_{II}, provided r is exterior to the shell. Thus, if $A(r')=A(r,r')$ represents the integrand of the surface integral in equation (27) or (28), and n' is normal into the shell at r'_I, on S_I, we have

$$A(r'_{II}) = \sum_{m=0}^{\infty} \frac{h^m}{m!} \frac{\partial^m}{\partial n^m} A(r'_I) \quad , \tag{29}$$

in which h is the thickness of the shell. Taking account of the opposite directions of the normals, on S_I and S_{II}, since the sign of $\int_S A(r')\, dS$ changes with the direction of the normal, we write

$$\int_{S'} A(r')\, dS = -\int_{S_I} A(r')\, dS + \int_{S_{II}} A(r')\, dS = \int_{S_I} \sum_{m=1}^{\infty} \frac{h^m}{m!} \frac{\partial^m A(r')}{\partial n'^m}\, dS, \tag{30}$$

the negative term, $-\int_{S_I} A(r')\, dS$, cancelling out the first term in the series (29). In equation (30) the normal is drawn from S_I into the shell.

Similarly, if r is exterior to the shell, we can express the integrand of the volume integral in equation (27) in terms of its values on S_I. Thus, with

n' the unit inward normal to S_I at r' we have

$$\int_V E(r')G(r,r')\,dV = \int_{S_I} dS \int_0^h E(r' + \eta' n')G(r, r' + \eta' n')\,d\eta'$$

$$= \int_{S_I} dS \int_0^h \sum_{m=0}^{\infty} \frac{\eta'^m}{m!} \frac{\partial^m}{\partial n'^m}\left[E(r')G(r,r')\right] d\eta' \qquad (31)$$

$$= \int_S \sum_{m=0}^{\infty} \frac{h^{m+1}}{(m+1)!} \frac{\partial^m}{\partial n'^m} (E(r')G(r,r')\,dS.$$

The volume integral in (28) can be written in a similar form, with E replaced by H.

Substituting from (30) and (31) into (27), with n denoting the inward-drawn normal to S_I, i.e., into the obstacle, we have

$$H(r) = E_o(r) + \frac{1}{4\pi} \int_{S_I} \sum_{m=0}^{\infty} \frac{h^{m+1}}{(m+1)!} \frac{\partial^m}{\partial n^m} \left[\frac{\partial}{\partial n}\left\{\left(\frac{\varepsilon'_b}{\varepsilon'_a} - 1\right)\left(\frac{\partial G}{\partial n} + 2GG_m\right) E^b_n n\right.\right.$$

$$\left. + \frac{i\omega}{c}\left(\mu_b - \mu_a\right) G(i_1 H_2^b - i_2 H_1^b) + \left(1 - \frac{\varepsilon'_b}{\varepsilon'_a}\right) G\left(i_1 \frac{\partial E^b_n}{\partial u_1} + i_2 \frac{\partial E^b_n}{\partial u_2}\right)\right\}$$

$$\left. + (k_b^2 - k_a^2)\, G\, E^b\right] dS. \qquad (32)$$

The summation in the integrand of equation (32) starts with $m = 0$, even though that of (30) starts with $m = 1$, because the power of h in (32) is $m + 1$. In this integrand the variable of integration is r' and extends over S_I. The equation is valid only if r is in region (a), i.e., external to the shell. Substituting from (30) into (28) and writing the volume integral in the same form as (31) with E replaced by H, we have

$$H(r) = H_o(r) + \frac{1}{4\pi} \int_{S_I} \sum_{m=0}^{\infty} \frac{h^{m+1}}{(m+1)!} \frac{\partial^m}{\partial n^m} \left[\frac{\partial}{\partial n}\left(\frac{\mu_b}{\mu_a} - 1\right)\left(\frac{\partial G}{\partial n} + 2\,GG_m\right) H^b_n\right.$$

$$- \frac{i\omega}{c}\left(\varepsilon'_b - \varepsilon'_a\right) G(i_1 E_2^b - i_2 E_1^b) + \left(1 - \frac{\mu_b}{\mu_a}\right) G\left(i_1 \frac{\partial H^b_n}{\partial u_1} + i_2 \frac{\partial H^b_n}{\partial u_2}\right)$$

$$\left. + (k_b^2 - k_a^2)\, G\, H^b\right] dS \qquad (33)$$

Inspection of equations (32) and (33) shows that if $h = 0$, $E(r) = E_0(r)$ and $H(r) = H_0(r)$, in accordance with the fact that the fields in the absence of the obstacle are $E_0(r)$ and $H_0(r)$. Since, on physical grounds, it is expected that the scattered fields will be small when the thickness h is small, it is natural to seek fields valid for shells of any thickness h in the form of a power series in h. We therefore try to find, in medium (a), solutions of the form:

$$E(r) = \sum_{\nu=0}^{\infty} h^{\nu} E_{(\nu)}(r), \quad H(r) = \sum_{\nu=0}^{\infty} h^{\nu} H_{(\nu)}(r) \text{ , where } E_{(\nu)}(r) \text{ and } H_{(\nu)}(r)$$

are functions to be determined.

Equations (32) and (33) can be used for the purpose of determining $E_{(\nu)}$ and $H_{(\nu)}$. In order to do this it is necessary to transform the field components and their derivatives in the integrands, which refer to medium (b), into expressions involving the corresponding quantities in medium (a). This can be accomplished by the use of the boundary conditions and Maxwell's equations, as was done for the field components and first derivatives in section 2. If after this has been done, the expansions for E(r) and H(r), valid in medium (a), are inserted into equations (32) and (33), these equations are identities in h, and hence like powers of h can be equated yielding equations for $E_{(\nu)}$ and $H_{(\nu)}$. When this is done we find that, because of the presence of the factor h^{m+1} in the integrands, the coefficient of h on the right-hand side of (32) or (33) can involve only those $E_{(q)}$ and $H_{(q)}$ for which $q < \nu$. Thus $E_{(\nu)}$ and $H_{(\nu)}$ can be found as surface integrals involving these $E_{(q)}$ and $H_{(q)}$. Since $E_{(o)} = E_o$ and $H_{(o)} = H_o$ are known, $E_{(\nu)}$ and $H_{(\nu)}$ can be found successively for $\nu = 0,1,2$, etc., by integration. The expressions for $E_{(1)}$ and $H_{(1)}$ are given below; for $\nu = 2,3$, etc., the expressions are similar.

In equations (34),(35) and (37) the field quantities with superscript b are defined in terms of the corresponding quantities with superscript a by equations(9), (10),(19),(23)-(26). Therefore $E^b_{(o)}$ and $H^b_{(o)}$ may be considered to be the extensions of $E^a_{(o)}$ and $H^a_{(o)}$ across the boundary surface by means of the boundary conditions. It is sufficient here, however, to consider them merely as abbreviations in order to simplify the integrands in equations (34), (35), and (37).

$$E_{(1)}(r)= \frac{1}{4\pi}\int_{S_I}\left[\frac{\partial}{\partial n}\left\{\left(\frac{\varepsilon'_b}{\varepsilon'_a}-1\right)\left(\frac{\partial G}{\partial n}+2GG_m\right)E^b_{n(o)}\ n\right.\right.$$

$$\left.+\frac{i\omega}{c}(\mu_b-\mu_a)\,G(i_1H^b_{2(o)}-i_2H^b_{1(o)})+\left(1-\frac{\varepsilon'_b}{\varepsilon'_a}\right)G\left(i_1\frac{\partial E^b_{n(o)}}{\partial \mu_1}+i_2\frac{\partial E^b_{n(o)}}{\partial \mu_2}\right)\right\}$$

$$+\left.(k_b^2-k_a^2)\,G\,E^b_{(o)}\right]dS, \qquad (34)$$

or

$$E_{(1)}(r)= \frac{1}{4\pi}\int\left\{n\left[\left(\frac{\varepsilon'_b}{\varepsilon'_a}-1\right)\left(\frac{\partial^2 G}{\partial n^2}+2G_m\frac{\partial G}{\partial n}\right)E^b_{n(o)}+\left(\frac{\varepsilon'_b}{\varepsilon'_a}-1\right)\left(\frac{\partial G}{\partial n}+2GG_m\right)\frac{\partial E^b_{n(o)}}{\partial n}\right.\right.$$

$$\left.+(k_b^2-k_a^2)\,G\,E^b_{n(o)}\right]+i_1\left[\frac{i\omega}{c}(\mu_b-\mu_a)\left(\frac{\partial G}{\partial n}H^b_{2(o)}+G\frac{\partial H^b_{2(o)}}{\partial n}\right)\right.$$

$$\left.+\left(1-\frac{\varepsilon'_b}{\varepsilon'_a}\right)\left(\frac{\partial G}{\partial n}\frac{\partial E^b_{b(o)}}{\partial \mu_1}+G\frac{\partial^2 E^b_{n(o)}}{\partial n\,\partial \mu_1}\right)+(k_b^2-k_a^2)\,G\,E^b_{1(o)}\right]$$

$$+i_2\left[-\frac{i\omega}{c}(\mu_b-\mu_a)\left(\frac{\partial G}{\partial n}H^b_{1(o)}+G\frac{\partial H^b_{1(o)}}{\partial n}\right)+\left(1-\frac{\varepsilon'_b}{\varepsilon'_a}\right)\left(\frac{\partial G}{\partial n}\frac{\partial E^b_{n(o)}}{\partial \mu_2}+G\frac{\partial^2 E^b_{n(o)}}{\partial n\,\partial \mu_2}\right)\right.$$

$$\left.\left.+(k_b^2-k_a^2)\,G\,E^b_{2(o)}\right]\right\}dS \qquad (35)$$

This expression for $E_{(1)}(r)$ is valid for r in medium (a). In order to evaluate it we must replace the field quantities and their derivatives expressed by means of superscript b by quantities to which they are equal in medium (a), by means of equations (9),(10),(19),(23)-(26). Thus, equation (35) becomes

$$E_{(1)}(r) = \frac{1}{4\pi} \int_{S_I} \left\{ n \left[E^a_{n(o)} \left(1 - \frac{\varepsilon'_a}{\varepsilon'_b}\right)\left(\frac{\partial^2 G}{\partial n^2} + 2G_m \frac{\partial G}{\partial n}\right) + \left(\frac{\varepsilon'_b}{\varepsilon'_a} - 1\right)\left(\frac{\partial G}{\partial n} + 2GG_m\right) \right.\right.$$

$$\left(\frac{\partial E^a_{n(o)}}{\partial n} + 2G_m E^a_{n(o)} \left[1 - \frac{\varepsilon'_a}{\varepsilon'_b}\right]\right) + (k_b^2 - k_a^2) \frac{\varepsilon'_a}{\varepsilon'_b} G E^a_{n(o)} \Big] + i_1 \left[\frac{i\omega}{c}(\mu_b - \mu_a)\right.$$

$$\left(\frac{\partial G}{\partial n} H^a_{2(o)} + G \frac{\partial H^a_{2(o)}}{\partial n} + \frac{i\omega}{c} G \left[\varepsilon'_b - \varepsilon'_a\right] E^a_{1(o)} + G\left[\frac{\mu_a}{\mu_b} - 1\right] \frac{\partial H^a_{n(o)}}{\partial \mu_2}\right)$$

$$+ \left(\frac{\varepsilon'_a}{\varepsilon'_b} - 1\right) \frac{\partial G}{\partial n} \frac{\partial E^a_{n(o)}}{\partial \mu_1} + G\left(1 - \frac{\varepsilon'_b}{\varepsilon'_a}\right)\left(\frac{\partial^2 E^a_{n(o)}}{\partial \mu_1 \partial n} + 2G_m \left[1 - \frac{\varepsilon'_a}{\varepsilon'_b}\right] \frac{\partial E^a_{n(o)}}{\partial \mu_1}\right)$$

$$\left. + (k_b^2 - k_a^2) G E^a_{1(o)} \right] + i_2 \left[- \frac{i\omega}{c}(\mu_b - \mu_a)\left(\frac{\partial G}{\partial n} H^a_{1(o)} + G \frac{\partial H^a_{1(o)}}{\partial n} - \frac{i\omega}{c} G \left[\varepsilon'_b - \varepsilon'_a\right] E^a_{2(o)}\right.\right.$$

$$+ G \left[\frac{\mu_a}{\mu_b} - 1\right] \frac{\partial H^a_{n(o)}}{\partial \mu_1}\Big) + \left(\frac{\varepsilon'_a}{\varepsilon'_b} - 1\right) \frac{\partial G}{\partial n} \frac{\partial E^a_{n(o)}}{\partial \mu_2} + G\left(1 - \frac{\varepsilon'_b}{\varepsilon'_a}\right)\left(\frac{\partial^2 E^a_{n(o)}}{\partial \mu_2 \partial n} + 2G_m\left[1 - \frac{\varepsilon'_a}{\varepsilon'_b}\right]\frac{\partial E^a_{n(o)}}{\partial \mu_2}\right.$$

$$\left.\left. + (k_b^2 - k_a^2) G E^a_{2(o)} \right]\right\} dS \qquad (36)$$

Similarly, from equation (33) we obtain, for $H_1(r)$,

$$H_1(r) = \frac{1}{4\pi} \int_{S_I} \left[\frac{\partial}{\partial n} \left\{ \left(\frac{\mu_b}{\mu_a} - 1\right)\left(\frac{\partial G}{\partial n} + 2G G_m\right) H^b_{n(o)} \right.\right. n$$

$$- \frac{i\omega}{c} (\varepsilon'_b - \varepsilon'_a) G(i_1 E^b_{2(o)} - i_2 E^b_{1(o)}) + \left(1 - \frac{\mu_b}{\mu_a}\right) G \left(i_1 \frac{\partial H^b_{n(o)}}{\partial \mu_1} + i_2 \frac{\partial H^b_{n(o)}}{\partial \mu_2}\right)$$

$$\left. + (k_b^2 - k_a^2) G H^b_{(o)} \right] dS. \qquad (37)$$

Carrying out the indicated differentiation and expressing all quantities in terms of medium (a),

$$H_{(1)}(r) = \frac{1}{4\pi}\int_{S_I}\Bigg\{ n\Bigg[\left(1-\frac{\mu_a}{\mu_b}\right)\left(\frac{\partial^2 G}{\partial n^2}+2G_m\frac{\partial G}{\partial n}\right) H^a_{n(o)}$$

$$+\left(\frac{\mu_b}{\mu_a}-1\right)\left(\frac{\partial G}{\partial n}+2GG_m\right)\left(\frac{\partial H^a_{n(o)}}{\partial n}+2G_m\left[1-\frac{\mu_a}{\mu_b}\right]H^a_{n(o)}\right)$$

$$+\left(k_b^2-k_a^2\right)\frac{\mu_a}{\mu_b}\,G\,H^a_{n(o)}\Bigg]+i_1\Bigg[-\frac{i\omega}{c}\left(\varepsilon'_b-\varepsilon'_a\right)\left(\frac{\partial G}{\partial n}E^a_{2(o)}+G\frac{\partial E^a_{2(o)}}{\partial n}\right.$$

$$-\frac{i\omega}{c}\,G\left[\mu_b-\mu_a\right]H^a_{1(o)}+G\left[\frac{\varepsilon'_a}{\varepsilon'_b}-1\right]\frac{\partial E^a_{n(o)}}{\partial \mu_2}\Big)+\left(\frac{\mu_a}{\mu_b}-1\right)\frac{\partial G}{\partial n}\,\frac{\partial H^a_{n(o)}}{\partial \mu_1}$$

$$+\left(1-\frac{\mu_b}{\mu_a}\right)G\left(\frac{\partial^2 H^a_{n(o)}}{\partial\mu_1\,\partial n}+2G_m\left[1-\frac{\mu_a}{\mu_b}\right]\frac{\partial H^a_{n(o)}}{\partial\mu_1}\right)+\left(k_b^2-k_a^2\right)G\,H^a_{1(o)}\Bigg]$$

$$+i_2\Bigg[\frac{i\omega}{c}\left(\varepsilon'_b-\varepsilon'_a\right)\left(\frac{\partial G}{\partial n}E^a_{1(o)}+G\frac{\partial E^a_{1(o)}}{\partial n}\right.+\frac{i\omega}{c}\,G\left[\mu_b-\mu_a\right]H^a_{2(o)}+G\left[\frac{\varepsilon'_a}{\varepsilon'_b}-1\right]\frac{\partial E^a_{n(o)}}{\partial\mu_1}\Big)$$

$$+\left(\frac{\mu_a}{\mu_b}-1\right)\frac{\partial G}{\partial n}\,\frac{\partial H^a_{n(o)}}{\partial\mu_2}+\left(1-\frac{\mu_b}{\mu_a}\right)G\left(\frac{\partial^2 H^a_{n(o)}}{\partial\mu_2\,\partial n}+2G_m\left[1-\frac{\mu_a}{\mu_b}\right]\frac{\partial H^a_{n(o)}}{\partial\mu_2}\right.$$

$$\left.+\left(k_b^2-k_a^2\right)G\;H^a_{2(o)}\Bigg]\right\}dS. \qquad (38)$$

Equations (36) and (38) represent a solution of the problem of scattering from a thin shell, correct to the first powers of h. This solution is valid at all points in medium (a), and is based on the assumption that there is no singularity in the scattered field. Solutions correct to higher powers of h can be obtained by

finding expressions for $E_{(2)}$, $H_{(2)}$, etc., by a process similar to that used to find the expressions for $E_{(1)}$ and $H_{(1)}$ in equations (36) and (38), as outlined in the paragraphs immediately following equation (33). One may also look for a solution in medium (b) in the form of a series in powers of h. The coefficients may be determined by a modification of the method used for finding those in medium (a). For the investigation of reflection, transmission, or scattering, however, only a solution in medium (a) is necessary, and as we have seen, in the case of a thin shell this can be obtained without an explicit expression for the solution in medium (b).

4. Evaluation of $E_{(1)}(r)$ and $H_{(1)}(r)$.

In order to evaluate $E_{(1)}(r)$ and $H_{(1)}(r)$ the form of the incident field and the expression for G must be known. The expression to be used for G is the Green's function (for the wave equation) for infinite space, namely $G(r,r') = \frac{e^{ik_a|r'-r|}}{|r'-r|}$ which evidently satisfies the conditions imposed in section 2. The expressions for the incident field will be written in the form

$$E_{(o)}(r) = \mathcal{E}(r)\, e^{ik_a|r - r_o|}, \quad H_{(o)}(r) = \mathcal{H}(r)\, e^{ik_a|r - r_o|}, \qquad (39)$$

where r_o is the center of the radiating system that produces the field. In particular, if the source is a dipole r_o is its geometric center; if it is a point dipole r_o is its position; in the case of plane waves r_o is at infinity in the direction from which the radiation comes. It is clear that any incident field can be written in the form (39). We use this form to emphasize explicitly our assumption that the main spatial variation of the field in the neighborhood of the obstacle is expressed by the exponential factor; that is, we assume that $\mathcal{E}(r)$ and $\mathcal{H}(r)$ are slowly varying in comparison with this factor near the obstacle. For plane waves $\mathcal{E}$ and $\mathcal{H}$ are constant and hence the assumption is obviously correct. For other sources it is the more accurate the greater the distance of the source..

It will be convenient to define the following angles:

γ = angle between $r'-r_o$ and n; thus $\cos\gamma = \frac{\partial|r'-r_o|}{\partial n}$

β = angle between $r'-r_o$ and u_1; thus $\cos\beta = \frac{\partial|r'-r_o|}{\partial u_1}$

δ = angle between $r'-r_o$ and u_2; thus $\cos\delta = \frac{\partial|r'-r_o|}{\partial u_2}$

α = angle between $r'-r$ and n; thus $\cos\alpha = \frac{\partial|r'-r|}{\partial n}$

In these definitions n represents the normal to S_I into the obstacle. With these definitions and the observation that the variations in G, $E_{(o)}$ and $H_{(o)}$ are mainly due to the exponential factor, all the derivatives appearing in equations (36) and (38) may be approximated in the following way. Using the chain rule we find

$$\left.\begin{aligned} &\frac{\partial G}{\partial n} = \left(ik_a - \frac{1}{|r'-r|}\right) G\cos\alpha \approx ik_a G\cos\alpha\ ; \\ &\frac{\partial E_o}{\partial n} = ik_a E_{(o)}\cos\gamma + \frac{\partial \mathcal{E}}{\partial n}\, e^{ik_a|r'-r_o|} \approx ik_a E_o \cos\gamma . \end{aligned}\right\} \tag{40}$$

Similarly

$$\left.\begin{aligned} &\frac{\partial E_{(o)}}{\partial u_1} \approx ik_a E_{(o)}\cos\beta\ ; \quad \frac{\partial E_{(o)}}{\partial u_2} \approx ik_a E_{(o)}\cos\delta\ ; \\ &\frac{\partial^2 G}{\partial n^2} \approx -k_a^2 G\cos^2\alpha\ ; \quad \frac{\partial^2 E(o)}{\partial u_1 \partial_n} = -k_a^2 E_{(o)}\cos\gamma\cos\beta\ ; \\ &\frac{\partial^2 E_{(o)}}{\partial u_2\ n} = -k_a^2 E_{(o)}\cos\gamma\cos\delta . \end{aligned}\right\}$$

The approximations for the derivatives of $H_{(o)}$ are similar to those of $E_{(o)}$.

Substituting from (39) and (40) in equations (36), the approximate value of $E_{(1)}(r)$ is found to be

$$E_{(1)}(r) \approx \frac{1}{4\pi}\int_{S_I}\Bigg\{ n\,\mathcal{E}_n\left[\left(1-\frac{\varepsilon_a'}{\varepsilon_b'}\right)\left(-k_a^2\cos^2\alpha + 2ik_a G_m\cos\alpha\right)\right.$$

$$+\left(\frac{\varepsilon_b'}{\varepsilon_a'}-1\right)\left(ik_a\cos\alpha + 2G_m\right)\left(ik_a\cos\gamma + 2G_m\left[1-\frac{\varepsilon_a'}{\varepsilon_b'}\right]\right) + \left(k_b^2-k_a^2\right)\frac{\varepsilon_a'}{\varepsilon_b'}$$

$$+i_1\left[\frac{i\omega}{c}\left(\mu_b-\mu_a\right)\left(ik_a H_2\cos\alpha + ik_a H_2\cos\gamma + \frac{i\omega}{c}\left[\varepsilon_b' - \varepsilon_a'\right]\mathcal{E}_1 + ik_a H_n\left(\frac{\mu_a}{\mu_b}-1\right)\cos\delta\right)\right.$$

$$-\left(\frac{\varepsilon_a'}{\varepsilon_b'}-1\right)k_a^2\,\mathcal{E}_n\cos\alpha\cos\beta + \left(1-\frac{\varepsilon_b'}{\varepsilon_a'}\right)\left(-k_a^2\,\mathcal{E}_n\cos\alpha\cos\beta + 2ik_a G_m\,\mathcal{E}_n\left(1-\frac{\varepsilon_a'}{\varepsilon_b'}\right)\cos\beta\right)$$

$$\left.\left.+\left(k_b^2-k_a^2\right)\mathcal{E}_1\right]+\right.$$

$$+ \; i_2 \left[-\frac{i\omega}{c}(\mu_b - \mu_a)(ik_a \mathcal{H}_1 \cos\alpha + ik_a \mathcal{H}_1 \cos\gamma - \frac{i\omega}{c}\left[\varepsilon_b' - \varepsilon_a'\right] \mathcal{E}_2 \right.$$

$$+ ik_a \mathcal{H}_n \left[\frac{\mu_a}{\mu_b} - 1\right] \cos\beta) - (\frac{\varepsilon_a'}{\varepsilon_b'} - 1) k_a^2 \mathcal{E}_n \cos\alpha \cos\delta + (1 - \frac{\varepsilon_b'}{\varepsilon_a'})(-k_a^2 \mathcal{E}_n \cos\gamma \cos\delta$$

$$\left. +2ik_a G_m \mathcal{E}_n \left[1 - \frac{\varepsilon_a'}{\varepsilon_b'}\right] \cos\delta) + (k_b^2 - k_a^2) \mathcal{E}_2 \right] \Bigg\} \frac{e^{ik_a(|r-r'| + |r'-r_o|)}}{|r - r'|} \; dS \qquad (41)$$

The corresponding expression for $H_{(1)}(r)$ can be obtained from equation (41) by interchanging $\mathcal{E}$ with $\mathcal{H}$ and ε' with $-\mu$.

The integral in equation (41) is of the type

$$I(r_1, r_2) = \int_S f(r') \; e^{i(k_1|r'-r_1| + k_2|r'-r_2|)} \; dS \qquad (42)$$

where $f(r') = \dfrac{\{\text{quantity in curly brackets}\}}{4\pi |r' - r|}$, $k_1 = k_2 = k_a$, $r_1 = r$, $r_2 = r_o$,

and $f(r')$ is slowly varying compared to the exponential factor. An approximate evaluation of the integral in equation (42) is given in Appendix I. In this evaluation the slowly varying factor $f(r')$ is taken out of the integrand and given an average value. Then the integral of the exponential factor is evaluated by a method similar to that of "stationary phase", applied to a double integral with finite limits of integration. The result of this evaluation is

$$I(r_1, r_2) = \frac{\bar{f}\, e^{iM'}}{J} \left[F_{\pm}(\xi_o'') - F_{\pm}(\xi_1'') \right] \left[F_{\pm}(\eta_o'') - F_{\pm}(\eta_1'') \right] . \qquad (43)$$

The symbols in the right-hand member of equation (43) have the following meanings (see Appendix I):

$\bar{f}$ = average value of f on S.

$F_{\pm}(x) = \int_0^x e^{\pm i \frac{\pi}{2} u^2} \, du$. In each bracket of equation (43) the signs agree and are chosen in accordance with the rule given after equation (67) of Appendix I.

ξ_o'', ξ_1'', η_o'', η_1'' are limits of integration, in suitably chosen units, corresponding to a rectangular region on the surface S which accounts for the main contribution to the integral.

$$M' = k_1 D_1 + k_2 D_2 + \text{other terms.}$$

In the application to equation (41), M' represents essentially the phase of the reflected field.

$$D_1 = \text{average value of } |r' - r_1|$$

$$D_2 = \text{average value of } |r' - r_2|$$

In the application these quantities represent average distances from the surface to the point of observation and the source, respectively.

J is given by equation (65) in Appendix I, and when $k_1 = k_2 = k_a$ for the application, it becomes

$$J = \frac{k_a}{\pi} \left\{ \left(\frac{\cos\alpha}{D_2} \right)^2 + \frac{1}{D_2} \left(\frac{1}{D_1} \left[2 - \sin^2\gamma - \sin^2\alpha + 2 \sin^2\alpha \sin^2\gamma \sin^2(\theta - \varphi) \right] \right.\right.$$

$$\left. - \left[\cos\alpha + \cos\gamma \right] \left[2G_m \cos^2\alpha + G_{11}(\theta) \sin^2\alpha \right] \right)$$

$$\left. + \left(\frac{\cos\gamma}{D_1} \right)^2 + G_g (\cos\alpha + \cos\gamma)^2 - \frac{1}{D_1} (\cos\alpha + \cos\gamma)(2G_m \cos^2\gamma + G_{11}(\varphi) \sin^2\gamma) \right\}^{\frac{1}{2}} \tag{44}$$

in which

θ = angle between u_1 and the plane containing the normal and r_1

φ = angle between u_1 and the plane containing the normal and r_2

G_m = mean curvature = $\frac{1}{2}(G_1 + G_2)$

G_g = Gaussian curvature = $G_1 G_2$

G_1, G_2 = principal curvatures.

$G_{11}(\theta)$ = curvature of the normal section of surface S determined by the plane making angle θ with u_1

$= G_1 \cos^2\theta + G_2 \sin^2\theta$

The quantities in J are determined at the point of S at which the phase of the integrand in equation (42) is stationary, if such a point exists. Otherwise, they are determined as described in Appendix I.

We apply the result of the evaluation in equation (43) to the integral in

equation (41) and obtain as an approximate value of E(r):

$$E(r) = E_{(o)}(r) + hE_{(1)}\, r$$

$$= E_{(o)}(r) + \left[\frac{e^{iF'}}{2}\left[F_{\pm}(x_o') - F_{\pm}(x_o'')\right]\left[F_{\pm}(y_o') - F_{\pm}(y_o'')\right]\right]\left[\frac{k_a \cos\alpha}{\pi D_2 J}\right]$$

$$\left[\frac{k_a h}{2}\left(n\mathcal{E}_n\left\{(1 - \frac{\varepsilon_a'}{\varepsilon_b'})(\frac{2iG_m}{k_a} - \cos\alpha) + (\frac{\varepsilon_b'}{\varepsilon_a'} - 1)(i + \frac{2G_m}{k_a \cos\alpha})(i\cos\gamma + \frac{2G_m}{k_a}\left[1 - \frac{\varepsilon_a'}{\varepsilon_b'}\right]\right)\right.\right.$$

$$\left. + (\frac{k_b^2}{k_a^2} - 1)\,\frac{\varepsilon_a'}{\varepsilon_b' \cos\alpha}\right\} + i_1\left\{\frac{i\omega}{k_a c}(\mu_b - \mu_a)(i\mathcal{H}_2\left[1 + \frac{\cos\gamma}{\cos\alpha}\right] + \frac{i\omega}{k_a c\cos\alpha}\left[\varepsilon_b' - \varepsilon_a'\right]\mathcal{E}_1 + i\,\frac{\cos\delta}{\cos\alpha}\mathcal{H}_n\left[\frac{\mu_a}{\mu_b} - 1\right]\right)\right.$$

$$- (\frac{\varepsilon_a'}{\varepsilon_b'} - 1)\cos\beta\,\mathcal{E}_n + (1 - \frac{\varepsilon_b'}{\varepsilon_a'})(-\frac{\cos\gamma}{\cos\alpha} + \frac{2iG_m}{k_a\cos\alpha}\left[1 - \frac{\varepsilon_a'}{\varepsilon_b'}\right]\mathcal{E}_n\cos\beta + (\frac{k_b^2}{k_a^2} - 1)\,\frac{\mathcal{E}_1}{\cos\alpha}\Big\}$$

$$+ i_2\left\{-\frac{i\omega}{k_a c}(\mu_b - \mu_a)(i\mathcal{H}_1\left[1 + \frac{\cos\gamma}{\cos\alpha}\right] - \frac{i\omega}{k_a c\cos\alpha}\left[\varepsilon_b' - \varepsilon_a'\right]\mathcal{E}_2 + i\,\frac{\cos\beta}{\cos\alpha}\mathcal{H}_n\left[\frac{\mu_a}{\mu_b} - 1\right]\right)\right.$$

$$- (\frac{\varepsilon_a'}{\varepsilon_b'} - 1)\cos\delta\,\mathcal{E}_n + (1 - \frac{\varepsilon_b'}{\varepsilon_a'})\left(-\frac{\cos\gamma}{\cos\alpha} + \frac{2iG_m}{k_a\cos\alpha}\left[1 - \frac{\varepsilon_a'}{\varepsilon_b'}\right]\right)\mathcal{E}_n\cos\delta$$

$$\left.\left. + (\frac{k_b^2}{k_a^2} - 1)\,\frac{\mathcal{E}_2}{\cos\alpha}\right\}\right] \qquad (45)$$

The corresponding expression for $H_{(1)}(r)$ can be obtained from equation (45) by interchanging $\mathcal{E}$ with $\mathcal{H}$ and ε' with $-\mu$.

Equation (45) and the corresponding equation for H(r) represent the electromagnetic fields in the presence of the shell, and therefore give the solution of the problem within the approximations indicated in the derivation. In principle, higher approximations to the fields could be obtained by use of equations (32) and (33). However, in order to use these equations to find the second approximation, for example, it is necessary to know the fields obtained by the first approximation on the obstacle surface, while the method used here to evaluate the integral in equation (41) is valid

only for observation points at some distance from the surface. Therefore equation (45) is not valid on the obstacle surface, and cannot be used for the determination of $\mathbf{E}_{(2)}$ and $H_{(2)}$. In fact, the evaluation of $\mathbf{E}_{(1)}$ and $H_{(1)}$ on S would require quite different methods from those we have used and that problem will not be dealt with here.

The remainder of the present paper will be devoted to the interpretation and discussion of the results obtained.

5. Properties of the Solution.

From equation (45) and its counterpart for H(r) it can be seen that the fields as approximated therein are composed of two parts: $\mathbf{E}_{(o)}$ and $H_{(o)}$, the incident fields, and $h\mathbf{E}_{(1)}$ and $hH_{(1)}$, the first order scattered, i.e., reflected, transmitted, or diffracted fields. The scattered field components are linear homogeneous expressions in the incident field components, and hence if the incident field is multiplied by a factor the scattered field is multiplied by the same factor. This was to be expected from the linear character of the problem. We wish to emphasize the fact (which is immediately evident) that each component of the scattered field is given as the product of three factors, each of which has physical significance. The factors are called the geometrical factor, the phase factor, and the reflectivity factor, respectively, and will now be examined in detail.

Geometrical factor. The geometrical factor G, the second factor enclosed in large brackets in equation (45), assumes the following form when the expression for J is inserted into it:

$$G = \left\{ 1 + D_2 \left(\frac{1}{D_1} \left[\frac{2 - \sin^2\gamma}{\cos^2\alpha} - \tan^2\alpha + 2\tan^2\alpha \sin^2\gamma \sin^2(\theta - \varphi) \right] \right.\right.$$
$$\left.\left. - (\cos\alpha + \cos\gamma)\left[2G_m + G_{11}(\theta)\tan^2\alpha \right] \right) + \left(\frac{\cos\gamma}{\cos\alpha} \frac{D_2}{D_1} \right)^2 + D_2^2 G_g \left(1 + \frac{\cos\gamma}{\cos\alpha} \right)^2$$
$$\left. - \frac{D_2}{D_1} \frac{\cos\alpha + \cos\gamma}{\cos^2\alpha} \left(2G_m \cos^2\gamma + G_{11}(\varphi) \sin^2\gamma \right) \right\}^{-\frac{1}{2}} \frac{\cos\alpha}{|\cos\alpha|} \qquad (46)$$

This factor is the same for all components of the scattered fields. It is called the geometrical factor for two reasons. In the first place, it depends only upon geometrical properties such as curvatures of the surface, distances of the source and observation points from the surface, and several angles. Secondly, the amplitude of the field reflected from a surface has been computed exactly according to geometrical optics.

(7) (12), and the ratio of the reflected amplitude to that of the field incident on the surface has been found to be precisely the above quantity G in directions of specular reflection (i.e. $\alpha = \gamma$, $\theta = \varphi$) and zero in other directions. (If, in the geometrical solution, one supposes only a fraction R^2 of the incident energy is reflected, then the ratio of the amplitude of the reflected field to that of the incident field is RG. By the use of ordinary geometrical optics, however, one cannot determine the value of R.*) This agreement leads to the interpretation of the factor G in the present solution as that part of the solution which describes the geometrical spreading of the reflected wave.

The same factor G appears in the solution by the Kirchhoff method of the problem of reflection, diffraction, and refraction of electromagnetic or acoustic waves by arbitrary curved surfaces, (7),(12). It also appears in the solution of the acoustic problem by geometrical optics and in the more exact treatment of acoustic reflection and transmission by thin shells (12). It follows, therefore, that whatever conclusions can be drawn from the consideration of this factor apply equally to all these cases.

In the important case of specular reflection ($\alpha = \gamma$, $\theta = \varphi$), G reduces to the form:

$$G = \left\{ 1 + 2D_2 \left[\frac{1}{D_1} - 2G_m \cos\gamma - G_{11} \sin\gamma \tan\gamma \right] + D_2^2 \left[\frac{1}{D_1^2} - \frac{4 G_m \cos\gamma}{D_1} + 4G_g - \frac{2G_{11} \sin\gamma \tan\gamma}{D_1} \right] \right\}^{-\frac{1}{2}} \frac{\cos\alpha}{|\cos\alpha|} . \quad (47)$$

In this case G, and therefore the reflected field, becomes infinite when

$$\frac{1}{D_2} = - \frac{1}{D_1} + 2G_m \cos\gamma + G_{11} \sin\gamma \tan\gamma \pm \Delta , \quad (48)$$

where $$\Delta^2 = (2G_m \cos\gamma + G_{11} \sin\gamma \tan\gamma)^2 - 4G_g \quad (49)$$

Points at which G is infinite are called conjugate points, and it can be seen from equation (48) that for each value of D_1 there are two such points along every direction of specular reflection if Δ is real and not zero. When these two coincide ($\Delta = 0$) a point image is formed. The conjugate points and point images are real or virtual according as the sign of D_2, as determined by equation (48) is positive or negative.

Point images occur only if $\Delta = 0$, and from examination of the expression for Δ we see that this can happen only if (1) $G_1 = G_2 = 0$, i.e., the surface point from which reflection occurs is a planar point, in which case $\Delta = 0$ for all directions;

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* It has been shown by R. K. Luneberg that the reflectivity factors of the geometrical solution for reflection from a single interface are just the well-known Fresnel formulas. See ref. (10) pp. 73-78.

or (2) $G_g > 0$, and if $|G_1| \geq |G_2|$ then $\theta = \frac{\pi}{2}$, $\cos\gamma = (\frac{G_2}{G_1})^{\frac{1}{2}}$, while if $|G_1| \leq |G_2|$ then $\theta = 0$, $\cos\gamma = (\frac{G_1}{G_2})^{\frac{1}{2}}$. It is interesting to observe that the condition for the occurrence of point images does not depend upon the distance D_1 of the source from the surface but does depend upon its direction. In the case of normal incidence $\cos\gamma = 1$, and hence a point image can occur only if $G_1 = G_2$, i.e., at a spherical point. For a point image the image distance is given by

$$\frac{1}{D_2} = -\frac{1}{D_1} + 2\sqrt{G_g} \quad , \tag{50}$$

as can be seen from equations (48) and (49). If the object distance D_1 is infinite, the image is called a focus, and D_2 is then called the focal length f.
Hence

$$f = \frac{1}{2\sqrt{G_g}} \quad . \tag{51}$$

Equations (50) and (51) yield the mirror law:

$$\frac{1}{D_1} + \frac{1}{D_2} = \frac{1}{f} \quad . \tag{52}$$

An examination of the phase factor, to be given later, will show that usually a change in phase of 180° occurs at point images and foci. A more detailed investigation of the geometrical and phase factors can yield further information about fields in the neighborhoods of foci.

The occurrence of such singularities as the above, where the field becomes infinite, appears to contradict one of the assumptions on which the solution was based, namely that the reflected field is regular. The solution in equation (41) is indeed regular, but in the evaluation of the integral a transformation of variables was used requiring a division by the Jacobian J, which is essentially $\frac{1}{G}$. At certain points - conjugate points and point images - J vanishes, but at such points the transformation is not valid and therefore the integral must be evaluated by other means. The present evaluation indicates, however, that the fields become large in the neighborhoods of such points. The integral can be evaluated approximately at these points by other means, and the results indicate that the fields there are large but finite.

In the case of specular reflection with normal incidence ($\alpha = \gamma = 0$) and $D_1 = D_2$, the observation point is at the source, and

$$G = \frac{1}{2}\left(1 + 2G_m D_1 + G_g D_1^2\right)^{-\frac{1}{2}}$$

$$= \frac{1}{2}\left(1 + \frac{D_1}{R_1}\right)^{-\frac{1}{2}}\left(1 + \frac{D_1}{R_2}\right)^{-\frac{1}{2}} \qquad (53)$$

Here R_1 and R_2 are the principal radii of curvature at the point on the surface from which reflection occurs, and the sign of G_m has been changed so that a curvature is positive if the center of curvature is on the opposite side of the surface from the source. When $D_1 \gg R_1$ add $D_1 \gg R_2$ equation (53) reduces to the well-known formula

$$G = \frac{1}{2D_1\sqrt{R_1 R_2}}$$

Other special cases of equation (53) are:

Plane ($R_1 = R_2 = \infty$) : $G = \frac{1}{2}$

Cylinder ($R_2 = \infty$) : $G = \frac{1}{2}\left(1 + \frac{D_1}{R_1}\right)^{-\frac{1}{2}}$

Sphere ($R_1 = R_2$) : $G = \frac{1}{2}\left(1 + \frac{D_1}{R_1}\right)^{-1}$

These special cases show the role of the geometrical factor in accounting for the spreading of the reflected field.

Finally, we consider the case of direct transmission, ($\alpha = \pi - \gamma$, $\theta = \varphi$). Here

$$G = \frac{-D_1}{D_1 + D_2} \qquad (54)$$

This result shows that G is independent of the curvature of the surface. Moreover, the spreading accounted for by this factor is the same as for a spherical wave diverging from the source.

The Phase Factor. The phase factor P, which is the first factor in large brackets in equation (45), like factor G. is the same for all components of the first order scattered field. It is given by

$$P = \frac{e^{iM'}}{2} \left[F_{\pm}(\xi_0'') - F_{\pm}(\xi_1'') \right] \left[F_{\pm}(\eta_0'') - F_{\pm}(\eta_1'') \right] \qquad (55)$$

This factor is called the phase factor for two reasons. In the first place, it depends upon the wave length of the incident field as well as upon geometrical properties, and oscillates as the wave length is varied, with its absolute value ranging between zero and approximately 2. Secondly, the problem of reflection and refraction of electromagnetic waves at arbitrarily curved surfaces and the analogous acoustic problem have been solved approximately by the Kirchhoff method, and these solutions all contain the above factor P as well as the geometrical factor G. The essential improvement of the Kirchhoff method over that of geometrical optics lies in the fact that the former takes account of the phase while the latter does not. In addition, P contains the factor $e^{ik_a (D_1 + D_2)}$ which describes the variation in the phase of the scattered wave due to the total distance traveled. For these reasons the factor P introduced by the Kirchhoff theory may be called the phase factor. This factor also occurs in the more general treatment of acoustic reflection and transmission [see ref. (12)].

As is shown at the end of Appendix I, for a fixed direction of the incident field the phase factor has its maximum absolute value in the direction of specular reflection for an obstacle that is symmetrical about the point of reflection. This is in accord with the experimental result that the reflected field is stronger in the specular direction than in non-specular directions, and indicates that the phase factor accounts for this difference. In ref. (12) it is shown that a slightly modified phase factor occurs in the solution of the refraction problem and has its maximum in the direction determined by Snell's law. As the wave length tends to zero ($k_a \to \infty$), the phase factor approaches zero in all directions except these, i.e., the direction of specular reflection in the case of reflection and the direction of Snell's law in the case of refraction, for which its absolute value approaches unity. This is shown in Appendix I. Thus, in the limit, our result yields the law of reflection which is usually taken as an assumption in geometrical optics (it is sometimes deduced from the assumption of the principle of least time).

The law of reflection has often been deduced from electromagnetic theory for the reflection of plane waves from an infinite plane; the present derivation also applies to curved surfaces and is not restricted to plane waves. Similarly, the modified phase factor, which is obtained when the Kirchhoff theory is applied to refraction, yields Snell's law for curved surfaces.

In Appendix I it is shown that the symbol $F_{\pm}(x)$ is written $F_{+}(x)$ or $F_{-}(x)$ according as A is positive or negative where A is the coefficient of x^2 in the phase of the integrand of equation (62). Similarly the sign in $F_{\pm}(y)$ is the same as that of C, the coefficient of y^2 in the phase of this integrand. By examining the expressions for A and C given in Appendix I it can be seen that on the specularly reflected ray through a point image, both A and C have the same sign on one side and both change sign on crossing the point image.

Thus the expression for the scattered field in equation (45) contains the Fresnel integrals F_{+} on one side and F_{-} on the other side of the point image. If the arguments of these functions are large, then $F_{\pm} \approx \frac{1}{\sqrt{2}} e^{\pm i\frac{\pi}{4}}$. Thus the product of Fresnel integrals changes from $\frac{1}{2} e^{+ i \frac{\pi}{2}}$ to $\frac{1}{2} e^{-i \frac{\pi}{2}}$ on crossing a point image along the reflected ray. Thus there is a change of phase of 180° in the scattered field on passing through any point image, in particular a focus, formed by reflection. It is shown in ref. (12) that the same considerations apply to point images formed by refraction.

The arguments of the Fresnel integrals are given by the following expressions:

$$\xi_0'' = \frac{\sqrt{2}\ \xi_0'}{\sqrt{\frac{\pi}{|A'|}}} \quad , \qquad \eta_0'' = \frac{\sqrt{2}\ \eta_0'}{\sqrt{\frac{\pi}{|C'|}}} \tag{56}$$

ξ_1'' and η_1'' are given by similar expressions in which ξ_0' and η_0' are replaced by ξ_1' and η_1', respectively. In these expressions ξ_0', ξ_1', η_0', η_1' are limits of integration over the rectangular region equivalent to the surface and referred to the point of specular reflection as origin. The denominators in these expressions can be given a simple interpretation in terms of the Fresnel zone concept, as follows.

Radiation from all points in a certain neighborhood of a given point on the surface arrives at a point r with phase differences less than π. Therefore the radiation from any point of this neighborhood reinforces that from other points of the neighborhood, rather than cancelling it. Such a neighborhood is called the first Fresnel zone around the given point. If the radiation comes from the point r_o and r' is a surface point, then the phase of the radiation reaching r from r' is $k_1 | r' - r| + k_1 | r' - r_o|$, while the phase of that from the given point is $k_1 D_1 + k_1 D_2$. For points on the boundary of the first Fresnel zone the difference of these values is $\pm \pi$. Using the coordinate system given in Appendix I with origin at the given point on the surface and the expressions for $k_1 | r' - r| + k_1 | r' - r_o|$ given in the exponent of the integrand in equation (62), we have as the equation of the boundary of the first Fresnel zone,

$$Ax^2 + Bxy + Cy^2 + Dx + Ey = \pm \pi \quad . \tag{57}$$

Rotating and translating the coordinate system

$$A'x'^2 + C'y'^2 = \pm \pi \quad ,$$

or

$$\left(\frac{x'}{\left| \frac{\pi}{A'} \right|^{1/2}} \right)^2 \pm \left(\frac{y'}{\left| \frac{\pi}{C'} \right|^{1/2}} \right)^2 = 1. \tag{58}$$

In equation (58) the sign is chosen the same as that of $A' C'$. Thus we see from equation (58) that the boundary of the first Fresnel zone is an ellipse if $A'C' > 0$, a hyperbola if $A'C' < 0$, and two parallel lines if either A' or C' is zero. From equation (58) it is seen that $\sqrt{\frac{\pi}{|A'|}}$ and $\sqrt{\frac{\pi}{|C'|}}$ are the half-dimensions of the first Fresnel zone around the given point. Thus the arguments of the Fresnel integrals are, except for the factor $\sqrt{2}$, the ratios of the half-dimensions of the surface to corresponding half-dimensions of the first Fresnel zone.

When the dimensions of the surface are large compared to those of the first Fresnel zone the arguments of the Fresnel integrals are large, and the integrals may be replaced by their asymptotic values, $\frac{1}{\sqrt{2}} e^{\pm i \frac{\pi}{4}}$. In this case the phase factor becomes $i e^{iM'}$ multiplied by 1 or -1, the choise of multiplier depending upon the

signs in the Fresnel integrals. From the form of M' it can then be seen that if medium (a) has zero conductivity, and k_a is consequently real, M' is real and therefore the phase factor has its absolute value equal to one. If the surface is much larger than the first Fresnel zone it will contain many Fresnel zones. This is usually the case unless the surface dimensions are small compared to the wave length, or one or both of the radii of curvature are infinite, i.e., the surface is cylindrical or plane.

When the surface is small compared to the first Fresnel zone the arguments of the Fresnel integrals are small, and these integrals may be approximated by the first terms in their Taylor expansions, which yields $F_{\pm}(x) \approx x$. Then

$$P = e^{iM'} \frac{(\xi'_o - \xi'_i)(\eta'_o - \eta'_i)}{\frac{\pi}{\sqrt{|A' C'|}}} \qquad (59)$$

From equation (59) we see that, apart from the exponential factor, P is equal to the quotient of area of the obstacle surface divided by the approximate area of the first Fresnel zone.

Since both A' and C' have the factor k_a, which is equal to $\frac{2\pi}{\lambda_a}$, where λ_a is the wave length in medium (a), if the conductivity of medium (a) is zero, P is inversely proportional to λ_a. The reflectivity factor also has the factor $k_a h$, and hence one sees that the scattered field is proportional to the obstacle volume (= area xh) and inversely proportional to λ_a^2. This is the analogue of the Rayleigh Scattering Law for electromagnetic waves. In particular, it is valid if $\lambda_a \gg$ obstacle dimensions, since then the first Fresnel zone is larger than the obstacle.

Reflectivity Factor. The scattered field is expressed in equation (45) in terms of components in the n, i_1, i_2 directions, where n, i_1, i_2 are mutually orthogonal unit vectors at the point of stationary phase on the surface S. Each of these components is the product of the geometrical and phase factors and a third factor which is called the reflectivity factor. Since this factor is not the same for all components, it is appropriate to speak of three reflectivity factors, one for each component. The factors are the coefficients of n, i_1, and i_2 in equation (45).

The reflectivity factors are linear and homogeneous in the components of the incident field. They depend upon the physical properties of both media, the thickness and curvature of the shell, and the angles of incidence and reflection. Thus it can be seen that these factors take account of the propagation within the shell. From an examination of the form of these factors it is seen that they are proportional to the amplitude of the incident field, and increase in absolute value as $|\epsilon'_a - \epsilon'_b|$ and/or $|\mu_a - \mu_b|$ increase. Moreover, they are proportional to $k_a h$ which, in the case when medium (a) has zero conductivity, is $\frac{2\pi h}{\lambda_a}$. Thus the scattered field increases with the ratio of the thickness to the wave length. This inverse dependence upon λ_a, in conjunction with the phase factor, leads to the electromagnetic analogue of the Rayleigh Scattering Law, as has been pointed out above. When reflection and diffraction problems are treated by the Kirchhoff method no reflectivity factor is obtained, and hence the Rayleigh Scattering Law cannot be deduced.

If the reflectivity factors are divided respectively by the corresponding field components, they agree with the exact coefficients obtained by R. K. Luneberg (13) for the reflection of plane waves from a thin, infinite, plane plate, except for additional terms proportional to the mean curvature of the surface. The polarization effects are therefore practically the same as for the plane case above. If the result in equation (45) is specialized to the case of plane waves incident on a plane shell, the geometrical factor and the phase factor both become unity, and the reflected and transmitted fields agree exactly with those obtained by Luneberg. This is shown in Appendix II.

The presence of the quantity $\cos \alpha$ in the denominator of some terms of the reflectivity factors appears to indicate that the scattered field becomes infinite as $\alpha \to \frac{\pi}{2}$. In this case, however, the method used to evaluate the integral is not

applicable. It should be noted that Luneberg's exact result, which remains finite as $\alpha \to \frac{\pi}{2}$, also contains $\cos \alpha$ in a denominator when it is expanded to apply to thin shells. This shows that the expansion is not valid as $\alpha \to \frac{\pi}{2}$, just as the method we have used for evaluating the integral fails in the same case.

6. Conclusion.

In this paper the general problem of scattering of an arbitrary incident electromagnetic field by an obstacle has been transformed to the solution of a pair of inhomogeneous linear integro-differential equations. For an obstacle which is a thin shell of constant thickness h we have given a formal procedure for obtaining solutions of these equations as power series in h. The lowest order term in this solution is the incident field, and an explicit expression for the next term has been found in the form of a surface integral. This integral has been evaluated approximately by an extension to surface integrals of the method of stationary phase. The resulting expressions for the first order scattered field give the field reflected, diffracted, or transmitted by a thin shell.

This solution has been compared with many results previously obtained by other methods, and in all cases the agreement has been satisfactory. Since the previous results have been extensively checked by experiment, the agreement furnishes some justification for our approximations. On the other hand, our result has been obtained directly from the fundamental equations of electromagnetic theory. The above comparisons therefore show that the results of the other methods are consequences of these equations, and also show under what conditions they are valid. Most of these comparisons, such as those with results of geometrical optics, the Kirchhoff method, and the Rayleigh Scattering Law, were discussed in section 5. Some of these results have not heretofore been deduced from electromagnetic theory.

The solution leads to a rather complete qualitative as well as quantitative understanding of the phenomena of reflection and transmission by thin shells. It also gives some insight into the relationships among the various approximation methods which have been used to treat these problems. Thus one finds that in geometrical optics the field reflected to any point comes entirely from one point, the stationary phase point, on the surface of the reflecting object. Since making the phase stationary is equivalent to making the optical path stationary, one sees

here the reason for the appearance of Fermat's "principle of least time" in geometrical optics. In the Kirchhoff theory the field specularly reflected to any point comes mainly from a neighborhood of the stationary point on the surface of the reflecting object. This accounts for the utility of the Fresnel zone concept in such problems. Finally, in the present paper, the main contribution to the field specularly reflected or transmitted to any point comes from a volume neighborhood of the stationary point, inside the shell, as well as from a neighborhood on the surface of the shell. This is the reason for the appearance, in the present solution, of the reflectivity factor which takes account of the electromagnetic properties of the shell material.

For reflection from thick shells additional terms in the series solutions would have to be evaluated. However, the necessary calculations become extremely complicated as was explained previously (see, also, comment in the next paragraph). An alternative method of dealing with this problem practically, which was found useful in acoustics, is the following. From the physical significance of the three factors in the solution of scattering by a thin shell it may be expected that the field scattered by any obstacle will also contain three such factors. This is further indicated by the fact that the phase and geometrical factors can also be obtained by the Kirchhoff method and the geometrical factor by geometrical optics, and both of these solutions apply to any obstacle - not only to a thin shell. Therefore one might expect that these two factors will have the forms already found for them, depending only upon the "front" surface of the obstacle. The reflectivity factor however, depending upon the thickness as well as the shape of the rear surface, may have quite a different form from that which it was found to have for thin shells. Since in this case the reflectivity factor is practically the same as the reflection coefficient given by R. K. Luneberg for the reflection of plane waves from a thin infinite plate, it is not unreasonable to assume that in the case of a thick shell it will be given by the corresponding expression for reflection of plane waves from a thick plate. Calculations based on these assumptions seem to agree with experiment in the acoustic case (12).

In order to provide a further test of the result of this paper, an exact solution for the reflection and transmission of spherical waves by a spherical shell with a dipole source at its center has been obtained explicitly (9). (It is remarkable that the solution of a problem apparently so simple leads to so complicated a result). This solution was specialized to the case of a thin shell and

compared to the result of the present paper when applied to this case. The agreement between the two solutions was satisfactory for all components except E_r, the radial component of electric field, and except near the source. It is at first sight surprising that any agreement at all was obtained, since in this case the reflected field has a singularity at the origin and therefore the result of this paper does not apply. The agreement may be understood by assuming that the result of this paper was applied to the problem with a slightly eccentric source, in which the reflected field does not have a singularity; the solution, except near the source, should be the same as that obtained with the source at the center. The error in the small longitudinal component E_r is due to the approximate evaluation of the integral in equqtion (41), to which E_r is very sensitive in this case. On the whole, the agreement is quite encouraging, indicating that the results of this paper may even be valid in cases when there are singularities in the scattered field, except near these singularities. The sensitive dependence of E_r is unimportant, since in most electromagnetic fields the longitudinal components are negligible.

Appendix I.

In this appendix we shall obtain an approximate evaluation of the integral

$$I(r_1,r_2) = \int_S f(r')\, e^{i(k_1 |r'-r_1| + k_2 |r'-r_2|)}\, dS' , \qquad (42)$$

where r' is a point on the surface S and r_1, r_2 are any two points many wavelengths away from the surface. This is the type of integral that it is necessary to evaluate in order to obtain the first order scattered field in equation (41). For the purposes of the present paper it would be satisfactory to have $k_1 = k_2$, and this specialization will be made when the result of this appendix is applied to the integral in equqtion (41). However, since an integral of the form given in equation (42) with $k_1 \neq k_2$ occurs in the treatment of refraction by the Kirchhoff method, it is convenient to retain k_1 and k_2, and thereby deal with this case also. It is assumed that $f(r')$ is slowly varying compared to the exponential factor. This is in accordance with observations about the expression for $f(r')$ in equation (41).

We construct a rectangular coordinate system with its origin at any point on the surface S, having the z-axis normal to the surface and the x,y-axes in directions of principal curvature of the surface at the origin. With this choice of coordinates, the equation of the surface is

$$z = ax^2 + by^2 + \ldots\ldots \qquad (60)$$

with no term in xy. If D_1 and D_2 are the respective distances from r_1 and r_2 to the origin, one may expand $r' - r_1$ for r' near the origin, as follows:

$$|r' - r_1| = \left[(x-x_1)^2+(y-y_1)^2 + (z-z_1)^2\right]^{\frac{1}{2}} \approx D_1 - \frac{x_1 x}{D_1} - \frac{y_1 y}{D_1}$$

$$+ \frac{1}{D_1}\left[\frac{1}{2} - az_1 - \frac{1}{2}\left(\frac{x_1}{D_1}\right)^2\right]x^2 + \frac{1}{D_1}\left[\frac{1}{2} - bz_1 - \frac{1}{2}\left(\frac{y_1}{D_1}\right)^2\right]y^2 - \frac{x_1 y_1}{D_1^3}\, xy$$

$$+ \text{terms of higher order.} \qquad (61)$$

In this expression x_1, y_1, z_1 are the coordinates of r_1. A similar expression may be obtained for $|r' - r_2|$ by replacing the subscript 1 by the subscript 2 in equation (61). In the expression so found x_2, y_2, z_2 are the coordinates of r_2.

If terms up to the second degree in the expansions for $|r' - r_1|$ and $|r' - r_2|$ are inserted into equation (42), one obtains the following approximate expressions for $I(r_1, r_2)$:

$$I(r_1, r_2) \approx \int_S f(r') e^{i\left[Ax^2 + Bxy + Cy^2 + Dx + Ey + M\right]} dS \qquad (62)$$

In this equation the following abbreviations have been used:

$$A = \frac{k_1}{D_1}\left[\frac{1}{2} - az_1 - \frac{1}{2}\left(\frac{x_1^2}{D_1^2}\right)\right] + \frac{k_2}{D_2}\left[\frac{1}{2} - az_2 - \frac{1}{2}\left(\frac{x_2^2}{D_2^2}\right)\right],$$

$$C = \frac{k_1}{D_1}\left[\frac{1}{2} - bz_1 - \frac{1}{2}\left(\frac{y_1^2}{D_1^2}\right)\right] + \frac{k_2}{D_2}\left[\frac{1}{2} - bz_2 - \frac{1}{2}\left(\frac{y_2^2}{D_2^2}\right)\right],$$

$$B = -\frac{k_1 x_1 y_1}{D_1^3} - \frac{k_2 x_2 y_2}{D_2^3}$$

$$D = -\frac{k_1 x_1}{D_1} - \frac{k_2 x_2}{D_2}, \qquad E = -\frac{k_1 y_1}{D_1} - \frac{k_2 y_2}{D_2}, \qquad M = k_1 D_1 + k_2 D_2$$

From equation (62) it can be seen that the phase of the integrand i.e. the exponent, will be stationary at the origin if, and only if, both D and E are zero. If there is a point on S for which both D and E are zero when that point is chosen as origin of coordinates, then the neighborhood of that point will give the main contribution to the integral. This is the principle of stationary phase, and is the more valid the larger are k_1 and k_2, which in the present case means the shorter are the wavelengths λ_1 and λ_2. The condition $D = E = 0$ is equivalent to the law of reflection when $k_1 = k_2$, and to Snell's law when $k_1 \neq k_2$. This means, in the case of reflection of radiation from a source at r_2 to a point r_1, that the main contribution comes from the neighborhood of that point r' on the surface, if any such point exists, for which the normal to S lies in the plane of r_1, r_2, r' and bisects the angle between $r' - r_1$ and $r' - r_2$. In the case of transmission by a shell, we have $k_1 = k_2$, and the condition $D = E = 0$ means that the main contribution to the wave transmitted to r_1 from the source at r_2 comes from the neighborhood of the point in which the straight line from r_1 to r_2 cuts the surface, if there is such a point.

Whenever such a stationary point can be found on the surface, it is to be chosen as the origin of coordinates. In case there is no such point, the principle of stationary phase indicates that the integral will be smaller in absolute value than it would be if there were one; this is borne out by the observation that non-specular reflection is much weaker than specular reflection, and will be explicitly shown by our result. In this case a point for which the phase is most nearly stationary may be chosen as origin. In any event, the expression for the phase in equation (62) can be considerably simplified by a translation and rotation of axes in the xy-plane. Thus we introduce the new variables x' and y', defined by the equations:

$$x' = x\cos\psi + y\sin\psi + H, \quad y' = -x\sin\psi + y\cos\psi + K,$$

where $\tan 2\psi = \frac{B}{A - C}$, and H and K are so chosen as to eliminate the first-degree terms in the transformed equation.

If we set

$$A' = A\cos^2\psi + B\cos\psi\sin\psi + C\sin^2\psi$$

$$C' = A\sin^2\psi - B\cos\psi\sin\psi + C\cos^2\psi$$

$$M' = A'H^2 + C'k^2 - D'H - E'k + M$$

where $D' = D\cos\psi + E\sin\psi$, $E' = E\cos\psi - D\sin\psi$,

then $H = \frac{D'}{2A'}$, $K = \frac{E'}{2C'}$. With these substitutions, equation (62) becomes

$$I(r_1, r_2) = \int_S f(r')\, e^{i(A'x'^2 + C'y'^2 + M)}\, dx'dy' \qquad (63)$$

It can be seen that the origin of the x', y' coordinate system is a point of stationary phase for the integrand of equation (63). If there exists a stationary phase point on the surface, in accordance with earlier remarks, it would have been chosen as the origin of x,y coordinates and hence also of the x',y' coordinates, the change of variables consisting merely of a rotation in this case. On the other hand, if there is no stationary phase point on the surface, the origin of the x,y coordinates is a non-stationary point, and the origin of the x',y' coordinates is then in the plane tangent to the surface at this point. In this case the origin of the x', y' coordinates will normally lie outside the range of

integration in equation (63), and there will then be no stationary point in the region of integration. It is therefore to be expected that the approximate value to be obtained for the integral -- which is small in this case -- will be less accurate than that obtained when there is a stationary point on the surface.

When there is a stationary point on S it is to be expected that the integral will be the greater the more nearly constant is the phase in its neighborhood. Thus if either A' or C', the second derivatives of the phase in equation (63), vanish, one would expect that the integral will be large in absolute value, and still larger when both vanish. The vanishing of these quantities depends upon the positions of r_1 and r_2 and on the curvatures of the surface at the stationary point, as can be seen from their definitions. When these quantities are such that neither A' or C' is zero, the following transformation can be made:

$$x'' = \sqrt{\frac{2|A'|}{\pi}}\, x' \qquad y'' = \sqrt{\frac{2|C'|}{\pi}}\, y' \tag{64}$$

Equation (63) then becomes

$$I(r_1,r_2) = \int_S f(r')e^{i\frac{\pi}{2}\left\{\frac{A'}{|A'|}\,x''^2 + \frac{C'}{|C'|}\,y''^2\right\} + iM'} \frac{1}{J}\,dx''\,dy'' \quad . \tag{65}$$

where J is the Jacobian of the transformation, and is given by

$$J = \frac{\partial(x'',y'')}{\partial(x',y')} = \frac{2}{\pi}|A'C'|^{\frac{1}{2}} = \frac{2}{\pi}\left|AC - \frac{B^2}{4}\right|^{\frac{1}{2}} .$$

$$= \frac{2}{\pi}\left\{\left[\frac{k_1}{D_1}\left(\frac{1}{2} - az_1 - \frac{1}{2}\frac{x_1^2}{D_1^2}\right) + \frac{k_2}{D_2}\left(\frac{1}{2} - az_2 - \frac{1}{2}\frac{x_2^2}{D_2^2}\right)\right]\left[\frac{k_1}{D_1}\left(\frac{1}{2} - bz - \frac{1}{2}\frac{y_1^2}{D_1^2}\right) + \frac{k_2}{D_2}\left(-bz - \frac{y_2^2}{D_2^2}\right)\right.\right.$$

$$\left.\left. - \frac{1}{4}\left(\frac{k_1 x_1 y_1}{D_1^3} + \frac{k_2 x_2 y_2}{D_2^3}\right)^2\right\}^{\frac{1}{2}} \tag{65'}$$

Let $k_1 = k_2 = k$, $\frac{x_1}{D_1} = \sin\gamma\cos\varphi$, $\frac{y_1}{D_1} = \sin\gamma\sin\varphi$, $\frac{z_1}{D_1} = \cos\gamma$;

$$\frac{x_2}{D_2} = \sin\alpha\cos\theta, \quad \frac{y_2}{D_2} = \sin\alpha\sin\theta , \quad \frac{z_2}{D_2} = \cos\alpha \; ; \; a + b = G_m,$$

$$4ab = G_g \; ; \; 2a\frac{x_1^2}{D_1^2} + 2b\frac{y_1^2}{D_1^2} = G_{11}(\theta)\sin^2\gamma .$$

Then J has the form given in equation (44), page 24.

From equation (65) and the first expression for J it is apparent that the integral is large when either or both of A' and C' are small. For values of r_1 such that A' or C' or both are zero, the integral in equation (65) becomes infinite. An investigation of such points, given in section 5, shows that they are the positions of conjugate points when only one of A' and C' vanishes, and point images or foci when both vanish. Actually, equation (65) is not valid in such cases, but it nevertheless indicates that the integral becomes large in absolute value near these points. Equation (63), which must be used at such points, shows that the integral remains finite there. The simultaneous vanishing of A' and C', which is necessary for the existence of point images or foci, occurs when certain relations involving the curvatures of the surface, the distances of the source and observation points, and the angle of incidence, are satisfied. At these points the fields are large. Point images or foci at which the fields are still greater may be obtained if higher derivatives of the phase, say third derivatives, vanish. Vanishing of these derivatives will require the surface and the incident ray to satisfy additional conditions which have not yet been investigated.

In order to evaluate the integral in equation (65) the limits of integration must be determined. The region over which the integral extends is the projection of the surface S on the x", y"-plane. If this region is a rectangle with sides parallel to the x", y"-axes, or a circle with center at the origin, the integral can be evaluated explicitly provided the slowly varying quantities f(r') and J are replaced by their average values. Any other region will be approximated by a rectangular region. This approximation might be expected to yield good results, since it agrees with a more accurate evaluation if the region is small, when the integral depends only upon its area and not on its shape; and if it is large, when the integral is usually independent of the shape. Thus, the limits of integration will be, in the rectangular approximation,*

$$\xi_o'' = \frac{\sqrt{2}\,\xi_o'}{\sqrt{\frac{\pi}{|A'|}}} \leq x \leq \xi_1'' = \frac{\sqrt{2}\,\xi_1'}{\sqrt{\frac{\pi}{|A'|}}}$$

$$\eta_o'' = \frac{\sqrt{2}\,\eta_o'}{\sqrt{\frac{\pi}{|C'|}}} \leq y \leq \eta_1'' = \frac{\sqrt{2}\,\eta_1'}{\sqrt{\frac{\pi}{|C'|}}} \qquad (66)$$

* If the boundary is a circle of radius ρ, in the x",y" coordinates, with center at the origin, and $A'/|A'| = C'/|C'| = \pm 1$, the integral yields

$$I(r_1,r_2) \approx \frac{\overline{f}\, e^{iM'}}{J} \cdot 2i(1 - e^{\pm i\pi\rho^2/2})$$

Here ξ_o', ξ_1', η_o', η_1' are the corresponding limits in the x', y' coordinates, and have the dimensions of length.

Equation (65) now becomes

$$I(r_1,r_2) = \frac{\mp e^{iM'}}{J}\left[F_\pm(\xi_o'') - F_\pm(\xi_1'')\right]\left[F_\pm(\eta_o'') - F_\pm(\eta_1'')\right] \qquad (67)$$

where $F_\pm(u)$ is the Fresnel integral $F_\pm(u) = \int_0^u e^{\pm i\frac{\pi}{2}v^2}\,dv$.

In the first bracket of equation (67) the sign is chosen the same as that of $\frac{A'}{|A'|}$, and in the second bracket the same as that of $\frac{C'}{|C'|}$. This is the approximate value of the integral in equation (42) which will be applied to the approximate evaluation of the integral for $E_{(1)}(r)$ in equation (41).

From the expression for $I(r_1,r_2)$ in equation (67) the following conclusions can be drawn. If the rectangle is symmetric about the origin in the x,y coordinates, with length L and width W,

then

$$\xi_o' = H - \frac{L}{2}, \qquad \xi_1' = H + \frac{L}{2},$$

$$\eta_o' = K - \frac{W}{2} \qquad \eta_1' = K + \frac{W}{2} \qquad (68)$$

In this case, if variations in the slowly changing quantity J are neglected, $I(r_1,r_2)$ has a maximum with respect to r_1 or r_2 when H = K = 0, which occurs if, and only if, D = E = 0. From the definitions of D and E this means

$$\frac{-k_1x_1}{D_1} = \frac{k_2x_2}{D_2}, \qquad -\frac{k_1y_1}{D_1} = \frac{k_2y_2}{D_2}. \qquad (69)$$

Equations (69) are equivalent to the law of reflection if $k_1 = k_2$, and to Snell's law if $k_1 \neq k_2$. In directions other than those given by these laws, either or both of H and K are different from zero. Since they are linear expressions in k_1 and k_2, they become infinite with k_1 and k_2. Then from equations (68), either ξ_o' and ξ_1', or η_o' and η_1', or both, become infinite, and the difference of at least one of the pairs of Fresnel integrals becomes zero. Thus, in the limit of zero wave length, in which case $k_1 = k_2 = \infty$, the integral is zero except in the directions given by geometrical optics.

Appendix II

In this appendix the results of the present paper for the reflected and transmitted fields will be specialized to the case of a plane wave incident on an infinite plane plate. They will then be compared with the exact solution of this problem given by R. K. Luneberg.

Since, in the case of an infinite plane, every point receives either specular reflection or direct transmission, one has $\alpha = \gamma$, $\theta = \varphi$ for reflection and $\alpha = \pi - \gamma$, $\theta = \varphi$ for transmission. The curvatures G_m and G_g are both zero at every point on the surface and therefore the directions u_1 and u_2 are arbitrary. For convenience, u_1 will be taken in the plane of incidence, positive in the direction of the reflected ray. By definition, the n-direction is normal to that surface of the plate nearest the source, and is positive into the plate. The u_2-direction is then determined.

For simplicity, an incident field with the electric vector normal to the plane of incidence will be assumed. The case in which the electric vector is parallel to the plane of incidence may be treated in exactly the same way by noticing that in that case the magnetic vector is normal to the plane of incidence. The following calculations and results then apply provided one interchanges E with H and ϵ' with $-\mu$. Thus a plane wave of arbitrary polarization may be dealt with since it can always be written as the sum of two fields, one with a normal electric vector and the other with a parallel electric vector.

A plane wave with its electric vector normal to the plane of incidence may be written

$$E_2 = A_o e^{ik_a|r-r_o|} , \quad E_1 = E_n = 0$$

$$H_n = A_o \sqrt{\frac{\epsilon'_a}{\mu_a}} \sin\gamma \, e^{ik_a|r-r_o|} ,$$

$$H_1 = -A_o \sqrt{\frac{\epsilon'_a}{\mu_a}} \cos\gamma \, e^{ik_a|r-r_o|} , \quad H_2 = 0$$

Here A_o is a constant scalar, the amplitude of the field. The source is located at r_o, which recedes to infinity along a line in the n, u_1-plane making the angle γ with the n-axis. Let the coordinates of r_o be z_o, x_o, 0 in the n,u_1,u_2-directions, respectively. Those of any other point r may be written as z,x,0, since the field and reflecting object do not vary with u_2 and therefore any point may be taken to

be in the plane $u_2 = 0$. One then has for the distance $|r - r_o|$

$$|r - r_o| = \left\{(z-z_o)^2 + (x - x_o)^2\right\}^{1/2} = (z - z_o)\cos\gamma + (x - x_o)\sin\gamma$$

The phase ζ of the plane wave is so adjusted that, as r_o goes to infinity, $\zeta - k_a z_o \cos\gamma - k_a x_o \sin\gamma$ is an even multiple of π. One then obtains the usual expression for a plane wave: $e^{ik_a(z\cos\gamma + x\sin\gamma)}$. We could, of course, have assumed this at the outset; the foregoing discussion justifies this form.

From the geometry, one finds that $\beta = \frac{\pi}{2} - \gamma$ and $\delta = \frac{\pi}{2}$.

Using the above data in equation (45) and the corresponding equation for H, one finds for both the reflected and transmitted fields at any point that $E_1 = E_n = H_2 = 0$. The phase factor becomes $i\, e^{ik_a(z\cos\gamma + x\sin\gamma)}$ for transmission and $i\, e^{ik_a(-z\cos\gamma + x\sin\gamma)}$ for reflection, since in both cases A and C are positive and the Fresnel integrals have their asymptotic values.

The geometrical factor becomes $-\frac{D_1}{D_1 + D_2} = -1$ for transmission, and

$\frac{D_1}{D_1 + D_2} = 1$ for reflection, since D_1 is infinite.

Using these expressions in equqtion (45) and the corresponding equation for H, the remaining reflected and transmitted field components are

Reflected

$$E_2 = A_o e^{-ik_a(z\cos\gamma - x\sin\gamma)}\left\{\frac{ik_a h}{2}\left[\frac{\mu_a^2 - \mu_b^2}{\mu_a\mu_b}\cos\gamma + \frac{\epsilon_b\mu_b - \epsilon_a\mu_a}{\epsilon_a\mu_b\cos\gamma}\right]\right\}$$

$$H_n = A_o\sqrt{\frac{\epsilon'_a}{\mu_a}}\sin\gamma\, e^{-ik_a(z\cos\gamma - x\sin\gamma)}\left\{\frac{ik_a h}{2}\left[\frac{\mu_a^2 - \mu_b^2}{\mu_a\mu_b}\cos\gamma + \frac{\epsilon'_b\mu_b - \epsilon'_a\mu_a}{\epsilon'_a\mu_b\cos\gamma}\right]\right\}$$

$$H_1 = A_o\sqrt{\frac{\epsilon'_a}{\mu_a}}\cos\gamma\, e^{-ik_a(z\cos\gamma - x\sin\gamma)}\left\{\frac{ik_a h}{2}\left[\frac{\mu_a^2 - \mu_b^2}{\mu_a\mu_b}\cos\gamma + \frac{\epsilon'_b\mu_b - \epsilon'_a\mu_a}{\epsilon'_a\mu_b\cos\gamma}\right]\right\}$$

Transmitted

$$E_2 = A_o e^{ik_a(z\cos\gamma + x\sin\gamma)}\left\{1 + \frac{ik_a h}{2}\left[\frac{(\mu_a - \mu_b)^2}{\mu_a\mu_b}\cos\gamma + \frac{\epsilon'_b\mu_b - \epsilon'_a\mu_a}{\epsilon'_a\mu_b\cos\gamma}\right]\right\}$$

$$H_n = A_o \sqrt{\frac{\varepsilon'_a}{\mu_a}} \sin\gamma \; e^{ik_a(z\cos\delta + x\sin\delta)} \left\{ 1 + \frac{ik_a h}{2} \left[\frac{(\mu'_a - \mu'_b)^2}{\mu_a \mu_b} \cos\delta + \frac{\varepsilon'_b \mu_b - \varepsilon'_a \mu_a}{\varepsilon'_a \mu_b \cos\delta} \right] \right\}$$

$$H_1 = -A_o \sqrt{\frac{\varepsilon'_a}{\mu_a}} \cos\gamma \, e^{ik_a(z\cos\delta + x\sin\delta)} \left\{ 1 + \frac{ik_a h}{2} \left[\frac{(\mu_a - \mu_b)^2}{\mu_a \cdot \mu_b} \cos\delta + \frac{\varepsilon'_b \mu_b - \varepsilon'_a \mu_a}{\varepsilon'_a \mu_b \cos\delta} \right] \right\}$$

The exact solution of reflection and transmission of plane waves by an infinite plate of any thickness has been given by R. K. Luneberg (11) p.48 . The amplitude of the reflected field is that of the incident field multiplied by the reflection coefficient R; similarly for the amplitude of the transmitted field and the transmission coefficient D. These coefficients are given below for the case of a parallel electric field. The sign of i has been changed in order to yeild agreement with our result since we use the factor $e^{-i\omega t}$ and Luneberg has used $e^{+i\omega t}$. Then one has

$$R_p = \frac{-i\left(\frac{M_a \varepsilon'_b}{M_b \varepsilon'_a} - \frac{M_b \varepsilon'_a}{M_a \varepsilon'_b}\right) \sin M_b T_b}{2\cos M_b T_b - i\left(\frac{M_a \varepsilon'_b}{M_b \varepsilon'_a} + \frac{M_b \varepsilon'_a}{M_a \varepsilon'_b}\right) \sin M_b T_b} \approx -\frac{i}{2}\left(\frac{M_a \varepsilon'_b}{M_b \varepsilon'_a} - \frac{M_b \varepsilon'_a}{M_a \varepsilon'_b}\right) M_b T_b$$

$$= \frac{ik_a h}{2} \left\{ \frac{\varepsilon'^2_a - \varepsilon'^2_b}{\varepsilon'_a \varepsilon'_b} \cos\delta + \frac{\varepsilon'_b \mu_b - \varepsilon'_a \mu_a}{\varepsilon'_b \mu_a \cos\delta} \right\}$$

$$D_p = \frac{2}{2\cos M_b T_b - i\left(\frac{M_a \varepsilon'_b}{M_b \varepsilon_a} + \frac{M_b \varepsilon'_a}{M_a \varepsilon'_b}\right) \sin M_b T_b} \approx 1 + \frac{i}{2}\left(\frac{M_a \varepsilon'_b}{M_b \varepsilon'_a} + \frac{M_b \varepsilon'_a}{M_a \varepsilon'_b}\right) M_b T_b$$

$$= 1 + \frac{ik_a h}{2} \left\{ \frac{\varepsilon'^2_a + \varepsilon'^2_b}{\varepsilon'_a \varepsilon'_b} \cos\delta + \frac{\varepsilon'_b \mu_b - \varepsilon'_a \mu_a}{\varepsilon'_b \mu_a \cos\delta} \right\}$$

where $M_a^2 = \varepsilon'_a \mu_a \cos^2\delta$, $M_b^2 = \varepsilon'_b \mu_b - \varepsilon'_a \mu_a \sin^{-2}\delta$, $T_b = \frac{h\omega}{c}$

The approximation is valid when $M_b T_b$ is small, which is the case if h is small enough. By reference to articles 4 and 5 of Luneberg's paper, one sees that the dependence of the reflected and transmitted fields on x,y,z is the same as that obtained in our solution. However, Luneberg's transmitted field has an extra factor $e^{ik_a h \cos \gamma}$. In order to compare the solutions, this must be expanded, yielding

$$D_p^* = D_p - ik_a h \cos \gamma = 1 + \frac{ik_a h}{2} \left\{ \frac{(\varepsilon'_a - \varepsilon'_b)^2}{\varepsilon'_a \varepsilon'_b} \cos \gamma + \frac{\varepsilon'_b \mu_b - \varepsilon'_a \mu_a}{\varepsilon'_b \mu_a \cos \gamma} \right\}$$

By comparing these solutions with the results of this paper, it is seen that the agreement is complete to the order of approximation retained. It must be noticed that Luneberg's result for the parallel electric field has been given; to compare it with our result for the normal electric field ε' and $-\mu$ must be interchanged.

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List of Symbols

Gaussian units are used throughout

∇^2 = Laplacian operator

$k = \frac{\omega^2}{c^2}\varepsilon'\mu$

$\varepsilon' = \varepsilon + \frac{4\pi i\sigma}{\omega}$

ω = angular frequency of field

ε = dielectric constant

μ = magnetic permeability

E = electric field (vector)

H = magnetic field (vector)

c = velocity of light in vacuum

r, r' = position vectors

$G(r,r')$ = Green's function defined on page 6

$|r - r'|$ = distance from r to r'

Q = any function of r, defined on page 6

V = three dimensional region, defined on p. 6

S = surface bounding V

n = normal vector to S into V

E_o, H_o = electric and magnetic fields in absence of obstacle, i.e. incident fields

$\bar{V}$ = three dimensional region in region (a)

$\bar{S}$ = surface of $\bar{V}$

S' = obstacle surface; i.e. surface of region (b)

σ_∞ = sphere of very large radius with center in region (b)

σ_r = small sphere with center at r

E^a, E^b = limits of $E(r)$ as r approaches the surface S' from media (a) and (b) respectively

H^a, H^b = limits of $H(r)$ as r approaches the surface S' from media (a) and (b) respectively

n, as subscript, indicates normal component of a vector at the surface S'

t indicates tangential component of a vector at the surface S'

u_1 = parameter which is constant along one family of lines of curvature on S'

u_2 = parameter which is constant along one family of lines of curvature on S'

$x(u_1\ u_2)$ = vector function of u_1, u_2 giving the surface S'

g_1, g_2 = metrical coefficients in element of arc length (page 10)

i_1, i_2 = unit vectors tangent to lines of curvature on S'

E_1, E_2 = components of E in directions of i_1, i_2 respectively

H_1, H_2 = components of H in directions of i_1, i_2 respectively

G_m = mean curvature at a point of S' (page 12)

S_I, S_{II} = parallel surfaces of the obstacle ; parts of S'

r_I, r'_{II} = points on S_I, S_{II} respectively

$A(r')$ = abbreviation for integrand of surface integral in eq. (27) or (28)

h = thickness of obstacle

η' = variable of integration ranging from 0 to h

n' = unit normal to S_I at r', into obstacle

ν = summation index, ranging from 0 to ∞ thru positive integers

$E_{(\nu)}(r)$, $H_{(\nu)}(r)$ = coefficients of h^ν in series expansion of E(r) and H(r) in medium (a). They are vector functions of r.

$\mathcal{E}(r), \mathcal{H}(r)$ = vector amplitudes of incident electric and magnetic fields

α, β, γ, δ = angles defined on p. 21

ξ', ξ'', η', η'' = limits of integration defined on p. 23 and p. 43

$F_{\pm}(x)$ = Fresnel integral defined on p. 23

G = Geometrical factor

P = Phase factor

The symbols in the following group are defined on page 24.

M'

D_1

D_2

J

θ, φ

G_g

G_1, G_2

G

Several symbols not appearing in the above list are defined in the appendices and used mainly in them.

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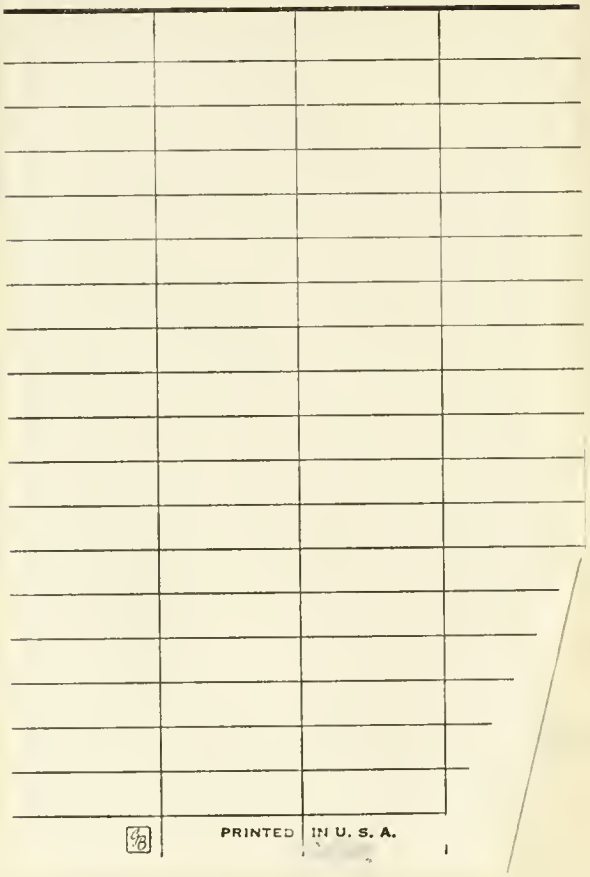

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